State and input simultaneous estimation for a class of nonlinear systems

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Abstract

This paper addresses the problem of estimating simultaneously the state and input of a class of nonlinear systems. Here, the systems nonlinear part comprises a Lipschitz nonlinear function with respect to the state and input, and a state-dependent unknown function including additive disturbance as well as uncertain/nonlinear/time-varying terms. Upon satisfying some conditions, the observer design problem can be solved via a Riccati inequality or a LMI-based technique with asymptotic estimation guaranteed. A numerical example is included for illustration.

Keywords: Observers; Nonlinear systems; State and input estimation

1. Introduction

The design of observers for nonlinear systems has recently received much attention. As mentioned in Zhu and Han (2002), there are generally two broad approaches for nonlinear observer design. In the first approach, the objective is to find a coordinate transformation so that the state estimation error dynamics are linear in the new coordinates and then linear techniques can be performed. Necessary and sufficient conditions have been established (Xiao & Gao, 1989; Marino, 1990) for the existence of such a coordinate transformation. In the second approach, methods have been developed to design observers for nonlinear systems without the need of the state transformation. Dynamic output feedback stabilization using high-gain observers has been studied for fully linearizable systems (Esfandiari & Khalil, 1992) and for systems with “input-to-state stable” inverse dynamics (Praly & Jiang, 1993). Observer-based output feedback has been developed for adaptive control of nonlinear systems (Krstic & Kokotovic, 1994). Adaptive observers have been proposed for special classes of nonlinear systems (Raghavan & Hedrick, 1994; Cho & Rajamani, 1997). The design of observers for nonlinear systems, where the estimation error decays irrespective of the input, has been reviewed and generalized in Besançon and Hammouri (1996).

For the class of global Lipschitz nonlinear systems, existence conditions have been established for full-order observers (Rajamani, 1998), and also for reduced-order observers (Zhu & Han, 2002). The design method is based on the solution of a Riccati equation. Observers that lead to an observer-error system with a state-independent error Lyapunov function have been proposed in Praly (2001) for lower triangular systems, i.e. systems whose dynamics are in a feedback form and where the nonlinear terms admit an incremental rate depending only on the measured output. For the class of systems with monotonic nonlinearities, by representing the observer error dynamics as the feedback interconnection of a linear system and a multivariable sector nonlinearity, where the linear part is required to be strictly positive real, Arcak and Kokotovic (2001a) proposed an observer design and robustness analysis using the circle criterion evaluated by LMI computations. The design is

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improved and necessary and sufficient conditions for its feasibility are established in Arcak and Kokotovic (2001b).

All the above-mentioned methods involve estimation of the state of nonlinear systems. There has been also an interest in estimation of both the state and the input of nonlinear systems. The problem is motivated in part by machine tool and manipulator applications. There are many situations where an input observer is required to estimate the cutting force of a machine tool or the exerting force/torque of a robotic system. In chaotic systems, one wishes to estimate not only the state for chaos synchronization but also the information signal input for secure communication (Liao & Huang, 1999). Compared with state estimation, less research has been carried out on estimating simultaneously the state of a system. In chaotic systems, one wishes to estimate not only the state of nonlinear systems. There has been also an interest in estimation of both the state and the input of nonlinear systems.

In Boutayeb et al. (2002), the problem of asymptotically estimating the system state and input has been addressed in Corless and Tu (1998), where the nonlinear part, expressed as a state-dependent and time varying function, is also the unknown input. Here, exact asymptotic estimation is not achieved, the system state and input can however be estimated to any desired degree of accuracy. More recently, nonlinear state space observer for simultaneously estimating the system state and input has been proposed in Boutayeb, Darouach, and Rafaralahy (2002). In their work, the nonlinear function is assumed to be Lipschitz with respect to the state and input.

Motivated by the work of Boutayeb et al. (2002), this paper is concerned with the design of an asymptotic observer to estimate both the state and input of a more general class of multiple-input multiple-output nonlinear systems. Our design technique has the following features: (i) the nonlinear function is not necessarily to be Lipschitz, rather it is treated more generally as a combination of a state-dependent unknown function including additive disturbance as well as uncertain/nonlinear/time-varying terms (Corless & Tu, 1998) and a nonlinear function, Lipschitz with respect to the state and input (Boutayeb et al., 2002); (ii) exact asymptotic convergence of the estimate can be achieved upon the satisfaction of some conditions. The organization of the paper is as follows. Following the introduction, Section 2 presents the problem statement. The main results are given in Section 3. Section 4 illustrates a numerical example. The proof of the main results is provided in Section 5. Finally, Section 6 concludes the paper.

2. Problem statement

Consider a class of nonlinear systems described by the following equations:
\[ \dot{x}(t) = Ax(t) + Bu(t) + f((x,u), y), \]  
\[ y(t) = Cx(t) + Du(t), \]
(1a)  
(1b)
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^r \) are, respectively, the state, unknown input and measured output. Matrices \( A, B, C \) and \( D \) are real constant and of appropriate dimensions. \( f(\ldots) \) is a real nonlinear vector function on \( \mathbb{R}^n \) and comprises two portions as follows:
\[ f((x,u), y) = f_L((x,u), y) + Wf_U((x,u), y), \]
(2)
where \( f_L((x,u), y) \in \mathbb{R}^n \) and \( f_U((x,u), y) \in \mathbb{R}^d \) are, respectively, known and unknown nonlinear vector field. Matrix \( W \) is real constant, assumed, without loss of generality, to have a full column rank of \( d \).

Our objective is to design an asymptotic observer from the measured output signals \( y(t) \) to estimate both the state \( x(t) \) and the unknown input \( u(t) \). The following assumptions are made throughout this paper:

**Assumption 1.** \( f_L(\ldots) \) is assumed to be Lipschitz in its first argument with a Lipschitz constant \( \gamma \), i.e.
\[ \| f_L(\tilde{\xi}, y) - f_L(\hat{\xi}, y) \| \leq \gamma \| \tilde{\xi} - \hat{\xi} \|, \quad \forall y, \]
(3)
where \( \tilde{\xi}(t) = [x(t) u(t)] \in \mathbb{R}^{n+m} \), \( \gamma \) is a positive real scalar, and \( \| \cdot \| \) denotes the norm symbol.

**Assumption 2.** Matrix \([DCW]\) has full column rank, i.e.
\[ \text{rank}[DCW] = m + d. \]
(4)

**Remark 1.** The system dynamics (1) are affected by Lipschitz nonlinearity and two arbitrary signals, the nonlinear terms that do not satisfy the Lipschitz condition and the unknown inputs. It is thus clear that the class of nonlinear systems considered in this paper is more general than those reported in the literature (Boutayeb et al., 2002; Corless & Tu, 1998).

**Remark 2.** A necessary condition for Assumption 2 to hold is that the number of outputs is least equal to the number of unknown inputs plus the size of the nonlinear terms which do not satisfy the Lipschitz condition, namely \( r \geq m + d \).

3. Main results

For the simplicity of presentation, let us introduce the following notations
\[ E = [I_n \quad 0_{n \times m}], \quad M = [A \quad B], \quad H = [C \quad D]. \]
(5)
The system described by (1)–(4) can then be expressed as
\[ E\dot{\tilde{x}}(t) = M\tilde{x}(t) + f_L(\tilde{\xi}, y) + Wf_U(\tilde{\xi}, y), \]
(6a)
\[ y(t) = H\tilde{x}(t), \]
(6b)
and the state and input estimation problem of (1) is now the problem of designing an observer for the generalized system (6) such that the estimate \( \hat{\xi}(t) \) converges asymptotically to \( \xi(t) \).
Consider now the following state observer for system (6):
\[
\dot{\hat{x}}(t) = N\hat{x}(t) + Ly(t) + T_{f_1}(\hat{\xi}, y),
\]
(7a)
\[
\dot{\hat{\xi}}(t) = \omega(t) + Qy(t),
\]
(7b)
where \(\dot{\hat{\xi}}(t)\) denotes the state estimation vector of \(\xi(t)\). Matrices \(N, L, T\) and \(Q\) are to be determined such that \(\dot{\hat{\xi}}(t)\) converges asymptotically to \(\hat{\xi}(t)\).

For the sake of simplicity, we introduce the following notation prior to stating the main results of this paper:
\[
\Phi = JM, \quad J = [I_{n+m} \quad 0]S^+ \begin{bmatrix} I_n \\ 0 \end{bmatrix},
\]
\[
S = \begin{bmatrix} E & W \\ H & 0_{r \times d} \end{bmatrix},
\]
(8)
\[
\Psi = GM, \quad G = (I_{n+r} - SS^+) \begin{bmatrix} I_n \\ 0 \end{bmatrix},
\]
(9)
\[
V = [I_{n+m} \quad 0]S^+ \begin{bmatrix} 0 \\ I_r \end{bmatrix}, \quad K = (I_{n+r} - SS^+) \begin{bmatrix} 0 \\ I_r \end{bmatrix},
\]
(10)
where \(S^+\) is a generalized inverse of \(S\), i.e. \(S^+ = (S^T S)^{-1} S^T\).

The following main theorem provides the theoretical basis for achieving our observer design goal.

**Theorem 1.** For system (6) satisfying Assumptions 1 and 2, the estimation error \(e(t) = \hat{\xi}(t) - \xi(t)\) of observer (7) converges asymptotically to zero if there exist matrices \(P = P^T > 0, X\) and \(Y\); and positive scalars \(\delta_1\) and \(\delta_2\) such that the following linear matrix inequality is satisfied:
\[
\begin{bmatrix}
\Phi^T P + \Psi^T Y^T - H^T X^T + P\Phi + YY^T - XH + \gamma^2(\delta_1 + \delta_2)I \\
J^T P \\
G^T Y^T
\end{bmatrix} < 0,
\]
(11)
where \(\gamma\) is the Lipschitz constant defined in (3). Furthermore, matrices \(T, Q, N\) and \(L\) of the observer are determined as
\[
T = J + ZG; \quad Z = P^{-1}Y,
\]
(12)
\[
Q = V + ZK,
\]
(13)
\[
N = \Phi + Z\Psi - FH; \quad F = P^{-1}X,
\]
(14)
\[
L = F + NQ.
\]
(15)

**Proof.** The proof of Theorem 1 will be given in Section 5.

The following corollary establishes the existence of the proposed observer for a special case of system (6) that is not subject to a Lipschitz nonlinearity \((f_1(x, u, y) = 0)\) (Corless & Tu, 1998).

**Corollary 1.** For system (6) that is not affected by Lipschitz nonlinearity, subject to Assumption 2 the estimation error \(e(t)\) of observer (7) converges asymptotically to zero if there exist matrices \(Z\) and \(F\) such that matrix \(N = (\Phi + Z\Psi - FH)\) is Hurwitz.

**Proof.** The proof of Corollary 1 will be given in Section 5.

**Remark 3.** The condition of Corollary 1 (i.e. matrix \(N\) is Hurwitz) indicates that the observer design can be accomplished with the determination of matrix \(Z \in \mathbb{R}^{(n+m) \times (n+r)}\) such that the system pair \((H, (\Phi + Z\Psi))\) is detectable, or of matrix \(F \in \mathbb{R}^{(n+m) \times r}\) such that the pair \(\{ -\Psi, (\Phi - FH)\}\) is detectable. Matrices \(T, Q\) and \(L\) are then determined as in (12), (13) and (15), respectively.

**Remark 4.** When \(G = 0\) (hence, \(\Psi = 0\)), one can choose \(Z = 0\) (hence, \(Y = 0\)) and the linear matrix inequality (11) is reduced simply to
\[
\begin{bmatrix}
\Phi^T P + P\Phi - H^T X^T - XH + \gamma^2(\delta_1 + \delta_2)I \\
J^T P \\
G^T Y^T
\end{bmatrix} < 0.
\]
(16)

**Remark 5.** In our approach the estimation error dynamics can be brought into the form (30) that is affected only by the Lipschitz nonlinearity. As noted in Rajamani (1998), asymptotic convergence of the estimation error will then necessitate that the Lipschitz constant \(\gamma\) should be less than some limit, in addition to the Hurwitz property of the observer error system’s matrix, which is \(N = (\Phi + Z\Psi - FH)\) in this paper. A smaller value of \(\gamma\) will therefore enable a better chance for the linear matrix inequality (11) to be feasible.

A computational algorithm is stated as follows.

**Design Algorithm.**

**Step 1:** Solve the LMI problem (11) by using the LMI toolbox.

**Step 2:** If \(P, X, Y, \delta_1\) and \(\delta_2\) satisfying (11) are found, then go to step 3.

**Step 3:** Obtain \(Z = P^{-1}Y\) and \(F = P^{-1}X\). Matrices \(T, Q, N\) and \(L\) are then given, respectively, by (12)–(15). The observer design is completed.

To illustrate the theoretical development and the design algorithm, a numerical example is included in the following section.
4. Numerical example

Consider a nonlinear system described by

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix} =
\begin{bmatrix}
-10 & 1 & 0 & 0 \\
-48.6 & -1.26 & 48.6 & 0 \\
0 & 0 & -22 & 1 \\
1.95 & 0 & -19.5 & -6
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix}
\]

\[+
\begin{bmatrix}
1 \\
0 \\
2 \\
0.5
\end{bmatrix}
\begin{bmatrix}
u(t)
\end{bmatrix}
\]

\[+
\begin{bmatrix}
0 \\
\alpha x_2(t)x_3(t) \\
0 \\
3.205\sin(x_3(t))
\end{bmatrix},
\]

\[y(t) =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix}
\]

and \(u(t)\) is an unknown input signal.

The objective of this example is to design an asymptotic observer to estimate simultaneously the states, \(x(t)\) and the unknown input signal, \(u(t)\). It is clear that the nonlinear function of this example is not fully Lipschitz due to the presence of the term \(\alpha x_2(t)x_3(t)\). On the other hand, the nonlinear function can be expressed into two portions defined in (2), as

\[
\begin{bmatrix}
0 \\
\alpha x_2(t)x_3(t) \\
0 \\
3.205\sin(x_3(t))
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
3.205\sin(x_3(t))
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
d(t),
\]

where \(d(t) = \alpha x_2(t)x_3(t)\), and the Lipschitz constant is \(\gamma = 3.205\).

Using (8)–(10), matrices \(S, J, \Phi, G, \Psi, V, \) and \(K\) can be easily obtained. Note that here \(G = 0\) and \(\Psi = 0\). With \(Y = 0\), the LMI problem yields the following:

\[
P =
\begin{bmatrix}
2.0731 & 0.1205 & 0.0509 & 0.2659 & 0.2543 \\
0.1205 & 1.7112 & -0.6647 & 0.5416 & 0.3853 \\
0.0509 & -0.6647 & 0.8233 & -0.6269 & -0.3101 \\
0.2659 & 0.5416 & -0.6269 & 0.8703 & 0.4124 \\
0.2543 & 0.3853 & -0.3101 & 0.4124 & 0.9842
\end{bmatrix},
\]

\[
T =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
Q =
\begin{bmatrix}
0 & 0 \\
-2 & 1 \\
0 & 0 \\
2 & 0
\end{bmatrix},
\]

\[
N =
\begin{bmatrix}
27.5118 & -6.8983 & -22 & 1 & 8.8575 \\
21.6775 & 1.7764 & -19.5 & -6 & 12.1402 \\
-2.2514 & 1.8242 & 0 & 0 & -9.3015 \\
0 & 0 & 1.1894 & 2.7132 & -2.2776
\end{bmatrix},
\]

\[
L =
\begin{bmatrix}
0 & 0 & 4 & 1 & 0 \\
1 & 2 & 0 & 0 & -2
\end{bmatrix}.\]

Accordingly, using (12)–(15), the following observer parameters are obtained:

For the sake of numerical simulation, the unknown input signal, \(u(t)\), is chosen as a sinusoidal signal, \(u(t) = 100\sin(9t)\). Superimposed in Fig. 1 are the time responses of the input signal \(u(t)\) and its estimate, \(\hat{u}(t)\). Fig. 2 shows typically the responses of a system state, \(x_1(t)\), and its estimate.

It is clear from the simulation that all the estimates of both the input and system state settle quickly to the actual responses of these signals. This verifies the theoretical results on estimation asymptotic convergence presented in Section 3.
5. Proof of the main results

Let us first introduce the following lemma.

**Lemma 1.** For observer (7) the estimate \( \hat{\xi}(t) \) will converge asymptotically to \( \xi(t) \) provided that the following conditions hold:

- **Condition 1:**
  \[
  \begin{cases}
  TE + QH = I_{n+m}, \\
  TW = 0.
  \end{cases}
  \]

- **Condition 2:**
  \[
  \begin{cases}
  N = TM - FH, \\
  F = L - NQ.
  \end{cases}
  \]

- **Condition 3:** The error \( e(t) \) determined by the observer error system

  \[
  \dot{e}(t) = Ne(t) + T\{f_L(e(t) + \xi(t), y(t)) - f_L(\hat{\xi}(t), y(t))\}
  \]

  converges asymptotically to zero for all \( \xi(t) \) and \( y(t) \).

**Proof.** Using (7b) and (6b), the estimation error \( e(t) = \hat{\xi}(t) - \xi(t) \) can be expressed as

\[
e(t) = \omega(t) + (QH - I_{n+m})\xi(t).
\]

Let \( T \) be an \((n+m) \times n\) matrix such that

\[
TE + QH = I_{n+m},
\]

then (18) becomes

\[
e(t) = \omega(t) - TE\xi(t).
\]

From (19), the error dynamics are

\[
\dot{e}(t) = \dot{\omega}(t) - TE\dot{\xi}(t).
\]

Substituting (7) and (6) into (20) and after some simple rearrangement, we obtain

\[
\dot{e}(t) = Ne(t) + (QH - L)\xi(t) - TWf_L(\xi(t), y)
\]

\[
+ T\{f_L(\hat{\xi}(t), y) - f_L(\xi(t), y)\}.
\]

(21)

From the first equation of Condition 1 of Lemma 1, one can easily verify that \( QH - L = N + (L - N)H - TM \). Thus, upon the satisfaction of both Conditions 1 and 2, Eq. (21) is reduced to

\[
\dot{e}(t) = Ne(t) + T\{f_L(\hat{\xi}(t), y) - f_L(\xi(t), y)\}
\]

\[
= Ne(t) + T\{f_L(e(t) + \xi(t), y(t)) - f_L(\xi(t), y(t))\}.
\]

Therefore, provided that Condition 3 of Lemma 1 is also met, then \( \hat{\xi}(t) \) is an asymptotic estimate of \( \xi(t) \). This completes the proof of Lemma 1. \( \square \)

**Proof of Theorem 1.** To prove Theorem 1, we shall establish all the conditions given in the above lemma. This can be done in two parts. The first part deals with the solvability of the set of matrix equations of Condition 1. The second part deals with the asymptotic convergence of the estimation error when incorporating Condition 2.

**Lemma 2.** Subject to Assumption 2, Condition 1 of Lemma 1 is always satisfied and matrices \( T, Q \) are given by

\[
T = J + ZG,
\]

\[
Q = V + ZK,
\]

where \( Z \) is an arbitrary matrix of dimension \((n+m) \times (n+r)\) and matrices \( J, G, V \) and \( K \) are as defined in (8)–(10).

**Proof.** Equations \( TE + QH = I_{n+m} \) and \( TW = 0 \) of Condition 1 can be combined into

\[
\begin{bmatrix}
T & Q
\end{bmatrix}S = \begin{bmatrix}
I_{n+m} & 0
\end{bmatrix},
\]

(24)

where matrix \( S \) is as defined in (8). A solution for the unknown matrix \( \begin{bmatrix} T & Q \end{bmatrix} \) in Eq. (24) exists iff (Rao & Mitra, 1971):

\[
\text{rank} \begin{bmatrix}
S & I_{n+m} & 0
\end{bmatrix} = \text{rank}(S).
\]

(25)

With \( W \) having a full column rank of \( d \), the left-hand side of (25) can be determined as

\[
\text{rank} \begin{bmatrix}
S & I_{n+m} & 0
\end{bmatrix} = \text{rank} \begin{bmatrix}
E & W \\ H & 0 \\ I_{n+m} & 0
\end{bmatrix}
\]

\[
= n + m + \text{rank} W = n + m + d.
\]

(26)
By using Assumption 2, the right-hand side of (25) can be expressed as

\[ \text{rank}(S) = \text{rank} \begin{bmatrix} I_n & 0 & W \\ C & D & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} I_n & 0 \\ -C & I_r \end{bmatrix} \begin{bmatrix} I_n & 0 & W \\ C & D & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} I_n & 0 & W \\ 0 & D & -CW \end{bmatrix} \]

\[ \Rightarrow \text{rank}(S) = n + \text{rank}[D - CW] = n + \text{rank}[D - CW] = n + m + d. \quad (27) \]

Thus, (25) is satisfied and hence, a solution to (24) exists and can be expressed as (Rao & Mitra, 1971)

\[ [T \; Q] = [I_{n+m} \; 0]S^+ + Z(I_{n+r} - SS^+). \quad (28) \]

From (28), matrices \( T \) and \( Q \) are easily derived. They are given, respectively, in (22) and (23), where matrices \( J, G, V' \) and \( K \) were defined in (8)–(10). This completes the proof of Lemma 2. ✔

Using (22), (8) and (9), matrix \( N \) defined in Condition 2 of Lemma 1 can now be expressed as

\[ N = TM - FH = \Phi + Z\Psi - FH. \quad (29) \]

Inserting this to the observer error system given in Condition 3 one obtains

\[
\dot{e}(t) = Ne(t) + T\{f_L(e(t) + \xi(t), y(t)) - f_L(\xi(t), y(t))\} \\
+ (\Phi + Z\Psi - FH)e(t) \\
+ (J + ZG)(f_L(e(t) + \xi(t), y(t)) \\
- f_L(\xi(t), y(t))). \quad (30)
\]

From (29) and (30), the estimation error, \( e(t) \), converges asymptotically to zero for all \( \xi(t) \) and \( y(t) \) provided that matrices \( Z \) and \( F \) are determined according to the following lemma.

**Lemma 3.** For observer (7), the estimation error \( e(t) \) converges asymptotically to zero if there exist matrices \( P = P^T > 0, Z, \) and \( F \); and positive scalars \( \delta_1 \) and \( \delta_2 \) such that the following Riccati inequality is satisfied:

\[ \Phi^TP + PF + \Psi^T\Psi P + PZ\Psi - H^TF^TP - PFH \\
+ \frac{1}{\delta_1}PJ^TP + \frac{1}{\delta_2}PZGG^TZ^TP + \gamma^2(\delta_1 + \delta_2)I < 0, \quad (31) \]

where \( \gamma \) is the Lipschitz constant defined in (3). Furthermore, the Riccati inequality (31) can be converted to the linear matrix inequality given in (11) (with respect to \( P, X, Y, \delta_1 \) and \( \delta_2 \)).

**Proof.** Consider a Lyapunov function

\[ V(e) = e^T(t)Pe(t), \quad (32) \]

where \( P = P^T > 0 \) is a positive-definite matrix. Taking its time derivative gives

\[ V'(e) = e^T(t)(P\Phi + PZ\Psi - FH + \Phi^TP \\
+ \Psi^T\Psi P - H^TF^TP)e(t) \\
+ e^T(t)(PZ + GZG^T)\tilde{f}_L + \tilde{f}_L^T(J^TP + \gamma^T\gamma^T)e(t), \quad (33) \]

where, for simplicity, \( f_L(e(t) + \xi(t), y(t)) - f_L(\xi(t), y(t)) \) is referred to as \( f \). Using the well-known matrix inequality \( z_1^T + z_2^T \leq \delta z_1^T + (1/\delta)z_2^T \), where \( \delta \) is a positive scalar and \( z_1 \) and \( z_2 \) are matrices of appropriate dimensions, the following inequality is obtained:

\[ \begin{align*}
\dot{e}^T(t)(PZ + GZG^T)\tilde{f}_L + \tilde{f}_L^T(J^TP + \gamma^T\gamma^T)e(t) & \leq \frac{1}{\delta_1}e^T(t)PJ^TPe(t) + \frac{\delta_1}{\delta_2}\tilde{f}_L^T\tilde{f}_L \\
& + \frac{1}{\delta_2}e^T(t)PZGG^T\gamma^T\gamma^Te(t) + \frac{\delta_2}{\delta_1}\tilde{f}_L^T\tilde{f}_L. \quad (34)
\end{align*} \]

Subject to Assumption 1, a substitution of (34) and (3) into (33) gives

\[ V(e) \leq e^T(t)Re(t) + (\delta_1 + \delta_2)\tilde{f}_L^T\tilde{f}_L \leq e^T(t)(R + \gamma^2(\delta_1 + \delta_2)I)e(t), \quad (35) \]

where

\[ R = \Phi^TP + P\Phi + \Psi^T\Psi P + PZ\Psi - H^TF^TP - PFH \\
+ \frac{1}{\delta_1}PJ^TP + \frac{1}{\delta_2}PZGG^T\gamma^T\gamma^T. \]

Accordingly, provided that (31) holds, then \( V(e) \) is negative definite. Hence \( e(t) \rightarrow 0 \) as \( t \rightarrow 0 \).

Now, in order for the Riccati inequality of the form \( R + \gamma^2(\delta_1 + \delta_2)I < 0 \) given in (31) to be solved in a computationally efficient manner using LMI techniques, available e.g., in MATLAB LMI Control toolbox (Gahinet, Nemirovski, Laub, & Chilali, 1995), the following one-to-one coordinate transformations are used:

\[ Y = PZ \quad \text{and} \quad X = PF. \quad (36) \]

Inequality (31) then becomes equivalently

\[ \begin{align*}
\Phi^TP + P\Phi + \Psi^T\Psi P + PZ\Psi - H^TF^TP - PFH \\
+ \frac{1}{\delta_1}PJ^TP + \frac{1}{\delta_2}PZGG^T\gamma^T\gamma^T \\
+ \frac{1}{\delta_1}PJJ^TP + \frac{1}{\delta_2}YGG^TY^T < 0.
\end{align*} \]

Using the Schur complement result (Cho & Rajamani, 1997), the above Riccati inequality can be easily converted to the linear matrix inequality (11). This completes the proof of Lemma 3. ✔

In summary, the proof of Theorem 1 is completed by noting that upon the satisfaction of the LMI (11), matrices
Z and F are derived from (36) and hence all the matrices of observer (7) are obtained as given in (12)–(15).

Proof of Corollary 1. The proof can be easily verified by noting that for the case when system (6) is not affected by Lipschitz nonlinearity \( f_l((x, u), y) = 0 \), subject to Assumption 2 the observer error system (30) becomes

\[
\dot{e}(t) = (\Phi + Z\Psi - FH)e(t).
\]

(37)

6. Conclusion

This paper has presented an approach to the design of observers for estimating simultaneously the state and unknown input for a class of nonlinear multi-input multi-output systems. It is interesting to note that the systems addressed in this paper are of a more general class than those reported in the literature. Sufficient conditions for asymptotic estimation convergence have been established. As a special case, the existence of such an observer when the system is not subject to Lipschitz nonlinearity is also discussed. The observer design can be completed in a computationally efficient manner via the use of an LMI-based algorithm. The design procedure and performance of the proposed method are illustrated through a numerical example.

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References


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