Observer-based control of multi-agent systems under decentralized information structure

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This paper addresses the problem of decentralized implementation of a global state feedback controller for multi-agent systems. The system is assumed to be under the constraint of a complete decentralized information structure. The decentralization of the control task is achieved through the construction of low-order decentralized functional observers with the purpose of generating the required corresponding control signal for each local control station. A design procedure is developed for obtaining an approximate solution to the design of the observers. Stability analysis is provided for the global system using the proposed observer-based approach. A numerical example is given to illustrate the design procedure and cases when the observers’ order increases from the lowest value.

1. Introduction

The task of effectively controlling multi-agent systems in the absence of communication channels is increasingly an interesting and challenging control problem. Centralized control of these systems requires the establishment of an extensive communication network for the transfer of information between each subsystem and the central station. This may impose cost and implementation problems which may prevent the realization of centralized controllers. Moreover, systems using explicit communication channels have, in most of the cases, a limited number of interconnected agents due to the communication requirements of the task set and the bandwidth of the channels. It is therefore desirable to devise a control strategy for such systems where the processing of information and the control task are shared among the decentralized controllers, provided that the overall system stability and performance are preserved. Obvious advantages of such control strategies are considerable savings in the cost of the information transfer network, and also the amenability of controller realization against system complexity. A considerable amount of research work has appeared in the control literature in an attempt to resolve the problems of decentralized control (see for example Wang and Davison (1973), Kobayashi et al. (1978), Kobayashi and Yoshikawa (1982), Jamshidi (1983), Trinh and Alden (1993), Ravi et al. (1995) and Chou and Cheng (2003), and references therein). The comment, made by Sandell et al. (1978) in a survey on the unresolved issue as to what decentralized structures would be suitable for a large-scale system, remains valid for the systems under the constraint of no communication between channels, which are addressed in this paper.

For autonomous systems, a decentralized framework, proposed by Stilwell and Bishop (2000), has shown to be applicable to the control of a platoon of underwater vehicles. Efficient methods of decentralized control design and implementation for a group of mobile robots are still an open research topic (Stilwell 2002). To facilitate implementation of a full state feedback control scheme for multi-agent systems, a linear functional observer scheme may be used to generate asymptotically an estimate of the globally produced control signals. The design of linear functional observers has been the focus of many researchers over the years. A number of procedures have been proposed for the design of linear functional state observers (see for example Sirisena (1979), Fairman and Gupta (1980), O’Reilly (1983), Tsui (1986), Trinh (1999) and Trinh and Ha...
Although linear functional observers have been developed for multivariable systems (Darouach 2000), little research has focused on their use, in a decentralized way, for multi-agent systems. This paper addresses the problem of controlling linear multivariable systems operating under a decentralized information structure, where no flow of information among the control stations takes place. For a control system consisting of \( N \) local control stations, the control input for the \( i \)th station is calculated from the information contained in its local input and output signals only.

Here, implementation of the global controller, obtained by using existing design methods, is accomplished by using \( N \) decentralized reduced-order linear functional observers to be designed. Significance of the proposed approach lies in the following features.

(i) The observers are completely decentralized in that each local control station uses locally available information only to generate the local control input signal and, hence, no information transfer among the local controllers is required;

(ii) Upon the satisfaction of some conditions, the overall closed-loop system stability and performance can be guaranteed;

(iii) The order of the each local observer can be selected from a lowest value.

The organization of the paper is as follows. The system in consideration is described in section 2, followed by the problem statement (section 3). Section 4 presents the main results including the design procedure and the stability analysis. Section 5 provides a numerical example. Finally, section 6 concludes the paper.

2. System description

Consider a linear time-invariant multivariable system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t),
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector and \( u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^r \) are the input and output vectors respectively. The matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{r \times n} \) are real constants.

Let \( N \) denotes the number of local control stations. Let the elements of the input vector \( u(t) \) and output vector \( y(t) \) be arranged so that

\[
\begin{align*}
u(t) &= \begin{bmatrix} u_1^T(t), & u_2^T(t), & \ldots, & u_N^T(t) \end{bmatrix}^T, \\
y(t) &= \begin{bmatrix} y_1^T(t), & y_2^T(t), & \ldots, & y_N^T(t) \end{bmatrix}^T,
\end{align*}
\]

where \( u_i(t) \in \mathbb{R}^m \) and \( y_i(t) \in \mathbb{R}^r \) \((i = 1, 2, \ldots, N)\) are the input and output vectors respectively of the \( i \)th control station. Accordingly, the system (1) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{N} B_i u_i(t), \\
y_i(t) &= C_i x(t), \quad i = 1, 2, \ldots, N,
\end{align*}
\]

where \( B_i \in \mathbb{R}^{m \times m} \) and \( C_i \in \mathbb{R}^{r \times n} \) are the submatrices of \( B \) and \( C \) respectively, determined according to equations (2).

Before proceeding with the main results let us introduce some assumptions.

**Assumption 1:** The triplet \((A, B, C)\) is controllable and observable.

**Assumption 2:** There exist no decentralized fixed modes (Wang and Davison 1973) associated with triplets \((B_i, A, C_i)\) or, if existing, they are assumed to be stable.

**Assumption 3:** Information available to the \( i \)th control station \( I_i(t) \) includes only the local output and control of the \( i \)th station:

\[
I_i(t) = \{ y_i(t), \ u_i(t) \}, \quad i = 1, 2, \ldots, N
\]

**Assumption 4:** A satisfactory global state feedback control law has been found of the form

\[
u(t) = Fx(t),
\]

where \( F \in \mathbb{R}^{m \times n} \) by using any standard state feedback control method to obtain the satisfaction of some system performance index.

Assumptions 1–4 are popular in linear systems. In order to satisfy assumption 2, it may be necessary to rearrange the partitions described in (3). Global controllability and observability guarantee that by suitably rearranging the control stations any decentralized fixed modes can be eliminated (Gong and Aldeen 1992). In the case when such a rearrangement entails the grouping of more than one control station, a limited amount of information transfer between the regrouped stations may be necessary. Assumption 3 represents a complete decentralized information structure of multi-agent systems considered in this paper.

3. Problem statement

Taking into account the constraint of the decentralized information structure (4), our objective is to design decentralized controllers of the form

\[
u_i(t) = f_i[I_i(t), t], \quad i = 1, 2, \ldots, N,
\]
using only information available to local control stations, that is $I_i(t)$, such that the multi-agent system (3) is stable with satisfactory performance as prescribed in the global control law (5). Note that (6) represents a decentralized control law for large-scale systems under a decentralized information structure, proposed by Kobayashi et al. (1978). To achieve the control objective, this paper seeks to reconstruct the global control (5) dynamically via decentralized linear functional observers that receive only $I_i(t)$ as their inputs. These observers are designed such that, upon the satisfaction of some conditions, the resulting local controls will reproduce asymptotically the global law (5) or at least guarantee the closed-loop stability for the multi-agent system (1–3).

First let us partition the global controller (5) as follows:

$$u(t) = F_1 x(t), \quad i = 1, 2, \ldots, N, \quad (7a)$$

or

$$u_i(t) = F_i x(t), \quad i = 1, 2, \ldots, N, \quad (7b)$$

where $F_i \in \mathbb{R}^{m_i \times n}$.

Equation (7b) implies that implementing the $i$th local controller requires the availability of $x(t)$, which may not be physically available for feedback. A decentralized control approach, which is based on the information structure (4), can overcome this problem. Such controllers, having the form of (6), are proposed here using decentralized dynamic systems described by

$$u_i(t) = F_i x(t) = (K_i L_i + W_i C_i) x(t) = K_i z_i(t) + W_i y_i(t), \quad (8a)$$

where

$$z_i(t) = E_i z_i(t) + L_i B_i u_i(t) + G_i y_i(t), \quad i = 1, 2, \ldots, N. \quad (8b)$$

In (8a) and (8b), $F_i = K_i L_i + W_i C_i$, $z_i(t) = L_i x(t) \in \mathbb{R}^{p_i}$ is the state vector of the dynamic system (8), and $K_i \in \mathbb{R}^{m_i \times p_i}$, $L_i \in \mathbb{R}^{n \times n}$, $W_i \in \mathbb{R}^{m_i \times r_i}$, $E_i \in \mathbb{R}^{p_i \times p_i}$ and $G_i \in \mathbb{R}^{p_i \times r_i}$ are real constant matrices to be determined.

Equation (8a) implies that $u_i(t)$ can now be implemented by using the local output $y_i(t)$ and a linear combination of global state $z_i(t) = L_i x(t)$, which can be generated from information (4), available at the $i$th control station only.

4. Main results

Let us assume, without loss of generality, that matrix $C_i$ has full row rank, that is $\text{rank}(C_i) = r_i$, and takes the canonical form

$$C_i = [I_{r_i} \ 0], \quad (9)$$

where $I_{r_i}$ is an identity matrix of dimension $r_i$. Note that any full rank matrix $C_i$ can always be transformed into (9) by the orthogonal transformation matrix

$$M_i = [C_i^T(C_i^T)^{-1} \ Q_i], \quad (10)$$

where $Q_i \in \mathbb{R}^{p_i \times (n-r_i)}$ is any basis of the null space of $C_i$, $\ker(C_i)$.

Let the global control input matrix $B$ be partitioned as

$$B = [B_i \ B_r], \quad (11)$$

where $B_i \in \mathbb{R}^{n \times (m-r_i)}$. Accordingly, (3) can be expressed as

$$x(t) = A x(t) + B_i u_i(t) + B_r u_r(t), \quad (12a)$$

$$y_i(t) = C_i x(t), \quad i = 1, 2, \ldots, N, \quad (12b)$$

where $u_i(t)$ contains $N - 1$ input vectors of the remaining $N - 1$ remote control stations.

Let an error vector $e_i(t)$ be defined as

$$e_i(t) = z_i(t) - L_i x(t), \quad i = 1, 2, \ldots, N. \quad (13)$$

By some simple manipulations, the following error equation is obtained:

$$\dot{e}_i(t) = \dot{z}_i(t) - L_i \dot{x}(t) = E_i \dot{z}_i(t) + L_i B_i u_i(t) + G_i y_i(t) - L_i A x(t) - L_i B_i u_i(t) + E_i e_i(t) + (G_i C_i - L_i A + E_i L_i)x(t) - L_i B_r u_r(t). \quad (14)$$

Equation (14) implies that the dynamic system (8b) can act as a decentralized linear functional observer for system (12), provided that matrix $E_i$ is chosen to be asymptotically stable and matrices $G_i$ and $L_i$ fulfill the constraints

$$G_i C_i - L_i A + E_i L_i = 0, \quad (15)$$

$$L_i B_r = 0, \quad (16)$$

$$F_i = K_i L_i + W_i C_i. \quad (17)$$
Matrix $E_i$ can be chosen according to the desired
dynamics of the observer to be constructed. There
are thus four unknown matrices $G_i$, $L_i$, $K_i$ and $W_i$ in
(15)–(17) to be solved for.

Using (9), then (15) and (17) can be expressed as
\[
G_i = (L_iA - E_iL_i) \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}, \quad (18a)
\]
\[
(L_iA - E_iL_i) \begin{bmatrix} 0 \\ I_{n-r_i} \end{bmatrix} = 0, \quad (18b)
\]
and
\[
W_i = (F_i - K_iL_i) \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}, \quad (19a)
\]
\[
(F_i - K_iL_i) \begin{bmatrix} 0 \\ I_{n-r_i} \end{bmatrix} = 0. \quad (19b)
\]

It is clear from (18a) and (19a) that matrices $G_i$ and $W_i$
can be directly derived, once matrices $K_i$ and $L_i$ are

\[
\Psi = \begin{bmatrix}
\alpha_{1,r_i+1}I_{p_i} & \alpha_{2,r_i+1}I_{p_i} & \cdots & \alpha_{r_i+1,r_i+1}I_{p_i} - E_i & \cdots & a_{n-1,r_i+1}I_{p_i} & a_{n,r_i+1}I_{p_i} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{1,n}I_{p_i} & \alpha_{2,n}I_{p_i} & \cdots & \alpha_{r_i+1,n}I_{p_i} & \cdots & a_{n-1,n}I_{p_i} & a_{n,n}I_{p_i}
\end{bmatrix},
\]

obtained. It remains therefore to solve (16), (18b) and
(19b) for matrices $K_i$ and $L_i$.

4.1. Design procedure

Let matrices $F_i$ and $L_i$ be partitioned as follows:
\[
F_i = [f_1 \ f_2 \ \cdots \ f_{r_i} \ | \ f_{r_i+1} \ f_{r_i+2} \ \cdots \ f_n], \quad (20)
\]
and
\[
L_i = [l_1 \ l_2 \ \cdots \ l_{r_i} \ | \ l_{r_i+1} \ l_{r_i+2} \ \cdots \ l_n], \quad (21)
\]
where
\[
f_{j} = \begin{bmatrix} f_{j,1} \\ f_{j,2} \\ \vdots \\ f_{j,m_i} \end{bmatrix} \in \mathbb{R}^{m_i} \quad \text{and} \quad l_{j} = \begin{bmatrix} l_{j,1} \\ l_{j,2} \\ \vdots \\ l_{j,p_i} \end{bmatrix} \in \mathbb{R}^{p_i} (j = 1, 2, \ldots, n)
\]
are the $j$th columns of matrices $F_i$ and $L_i$ respectively.

Incorporating (20) and (21) into (19b), after some
rearrangement, gives the matrix–vector equation
\[
\Phi l = f, \quad (22a)
\]
where

\[
\Phi = [0_{m(n-r_i) \times p_{r_i}} \ \Omega], \quad (22b)
\]
\[
\Omega = \text{diag}(K_i) \in \mathbb{R}^{m(n-r_i) \times p(n-r_i)}, \quad (22c)
\]
\[
l = [l_1^T \ l_2^T \ \cdots \ l_n^T]^T \in \mathbb{R}^{p_n}, \quad (22d)
\]
\[
f = [f_{r_i+1}^T \ f_{r_i+2}^T \ \cdots \ f_n^T]^T \in \mathbb{R}^{m(n-r_i)}, \quad (22e)
\]

In (22b), $0_{m(n-r_i) \times p_{r_i}}$ is a zero matrix of dimension
$m_i(n-r_i) \times p_{r_i}$.

Let us now consider (18b). As commented earlier,
$E_i \in \mathbb{R}^{m \times p}$ can be chosen to have a desired stable eigen-
structure. By some simple rearrangement, (18b) can be
put in a matrix–vector form as
\[
\Psi l = 0, \quad (23a)
\]
where matrix $\Psi \in \mathbb{R}^{p(n-r_i) \times p_n}$ is given by
\[
\Psi = \begin{bmatrix}
\alpha_{1,r_i+1}I_{p_i} & \alpha_{2,r_i+1}I_{p_i} & \cdots & \alpha_{r_i+1,r_i+1}I_{p_i} - E_i & \cdots & a_{n-1,r_i+1}I_{p_i} & a_{n,r_i+1}I_{p_i} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{1,n}I_{p_i} & \alpha_{2,n}I_{p_i} & \cdots & \alpha_{r_i+1,n}I_{p_i} & \cdots & a_{n-1,n}I_{p_i} & a_{n,n}I_{p_i}
\end{bmatrix},
\]
and $a_{j,k}$ denotes the $(j, k)$ element of matrix $A$.

Consider now (16). Similarly, it can be put in a
matrix–vector form as
\[
\Theta l = 0, \quad (24a)
\]
where
\[
\Theta = \begin{bmatrix}
b_{1,1}I_{p_i} & b_{2,1}I_{p_i} & \cdots & b_{n,1}I_{p_i} \\
b_{1,2}I_{p_i} & b_{2,2}I_{p_i} & \cdots & b_{n,2}I_{p_i} \\
\vdots & \vdots & \ddots & \vdots \\
b_{1,m_i-m_i}I_{p_i} & b_{2,m_i-m_i}I_{p_i} & \cdots & b_{n,m_i-m_i}I_{p_i}
\end{bmatrix} \in \mathbb{R}^{m(n-m_i) \times p_{n}},
\]
and $b_{j,k}$ denotes the $(j, k)$ element of matrix $B_r$.

The problem is thus concluded to be that of finding a
feasible solution to equation
\[
\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} l = \begin{bmatrix} f \\ 0 \end{bmatrix}. \quad (25)
\]

Remark 1: For a given $K_i$, (25) consists of $m_i(n-r_i) + p_i(n-r_i) + p(m-m_i)$ linear simultaneous equations
with $p, m$ unknowns. Consequently, the $i$th decentralized control law (8a) can be exactly implemented if $m_i(n - r_i) + p_i(n - r_i) + p_i(m_i - m) \leq p_i m$, or $r_i + m_i > m$ and the observer order $p_i$ chosen such that $p_i \geq m_i(n - r_i)/(r_i + m_i - m)$.

If an exact solution to (25) is obtained, then with $E$, selected to be Hurwitz, error vectors $e(t)$ asymptotically approach zero. The local control laws (8a) will therefore reproduce asymptotically the global control (5). However, exact solutions to (25) may not always be found to satisfy (22)–(24), especially for low orders $p_i$ of the observers (8). In the following, an alternative procedure for solving matrices $K_i$ and $L_i$ is proposed. The procedure involves the formulation and solution of an optimization problem, which will minimize the norm of the error between the two sides of (22) and (23). It will be shown later that the error norm of these two equations will determine the overall closed-loop stability of the system.

Let us now consider (24) and assume that matrix $\Theta$ has full row rank, that is $\text{rank}(\Theta) = p(m - m_i)$ (this condition corresponds to the requirement that $B_i$ has full column rank). Thus, there exists an orthogonal connection matrix $T_i \in \mathbb{R}^{n \times p_i m}$, where $T_i^T = (T_i)^{-1}$, such that (24a) can be transformed into
\[
\tilde{\Phi}l = [\tilde{\Phi}_1 \quad \tilde{\Phi}_2][\tilde{l}_1 \quad \tilde{l}_2] = 0, \quad \text{(26a)}
\]
where
\[
\tilde{\Phi} = \Theta T_i^T, \quad \text{(26b)}
\]
\[
\tilde{l} = T_i l, \quad \text{(26c)}
\]
matrix $\tilde{\Phi}_1 \in \mathbb{R}^{p_i m_i \times p_i (m - m_i)}$ is invertible, $\tilde{\Phi}_2 \in \mathbb{R}^{p_i(m - m_i) \times p_i (n - m_i - m)}$, $\tilde{l}_1 \in \mathbb{R}^{p_i (m - m_i)}$ and $\tilde{l}_2 \in \mathbb{R}^{p_i (n - m_i - m)}$.

The following equation is directly derived from (26a):
\[
\tilde{l}_1 = \mu \tilde{l}_2, \quad \text{(27a)}
\]
where
\[
\mu = - (\tilde{\Phi}_1)^{-1} \tilde{\Phi}_2. \quad \text{(27b)}
\]
Rearranging (22a) and (23a) in accordance with (26) gives
\[
\Phi l = (\Phi T_i^T)(T_i l) = \tilde{\Phi}l = [\tilde{\Phi}_1 \quad \tilde{\Phi}_2][\tilde{l}_1 \quad \tilde{l}_2] = f, \quad \text{(28)}
\]
and
\[
\Psi l = (\Psi T_i^T)(T_i l) = \tilde{\Psi}l = [\tilde{\Psi}_1 \quad \tilde{\Psi}_2][\tilde{l}_1 \quad \tilde{l}_2] = 0. \quad \text{(29)}
\]
Finally, by substituting (27a) into (28) and (29) and after some manipulations, the following algebraic equation is obtained:
\[
\beta l_2 = \gamma, \quad \text{(30a)}
\]
where
\[
\beta = \begin{bmatrix} \Phi_1 \mu + \Phi_2 \\ \Psi_1 \mu + \Psi_2 \end{bmatrix} \in \mathbb{R}^{m_i (n - r_i) + p_i (n - r_i) \times p_i (n - m_i + m)}, \quad \text{(30b)}
\]
and
\[
\gamma = \begin{bmatrix} f \\ 0 \end{bmatrix} \in \mathbb{R}^{m_i (n - r_i) + p_i (n - r_i)}. \quad \text{(30c)}
\]
Solving (25) is now equivalent to solving (30a), an exact solution of which, in general, does not always exist. However, as matrix $\beta$ contains $m_i \times p_i$ elements of matrix $K_i$, one can solve approximately (30a) by minimizing the error norm
\[
\delta(K_i) = \| \beta \beta^+ \gamma - \gamma \|, \quad \text{(31)}
\]
where $\beta^+ = (\beta^T \beta)^{-1} \beta^T$ is the Moore–Penrose pseudo-inverse of $\beta$, provided that $\beta$ has a full column rank. Vector $\tilde{l}_2$ is then derived from
\[
\tilde{l}_2 = \beta^+ \gamma. \quad \text{(32)}
\]
The above minimization problem (31) can be solved by searching within a bounded range for $m_i \times p_i$ elements of matrix $K_i$ regarding the $i$th local control station, where matrix $K_i$ has full row rank. The search, for each local control station, involves a small number of parameters $m_i p_i$ and can be done by using the function constr for finding the constrained minimum of a function, available in the MATLAB Optimization Toolbox. Having obtained matrix $K_i$ and vector $\tilde{l}_2$, vector $\tilde{l}_1$ is then derived from (27) and, hence, $l$ is obtained from $l = T_i \tilde{l}$. Consequently, matrices $L_i$, $G_i$ and $W_i$ can be straightforwardly derived from (21), (18a) and (19a) respectively.

Remark 2: For the case where matrix $\beta$ does not have full column rank, vector $\tilde{l}_2$ can be solved for by using a matrix singular value decomposition as outlined below.

Let matrix $\beta$ be decomposed as
\[
\beta = USV = [U_1 \quad U_2] \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} V, \quad \text{(33)}
\]
where matrices $S_1$ is a non-singular diagonal matrix containing the singular values. Substituting (33) into (30a), after some simple manipulations, gives

$$\tilde{p}_2 = H\eta = [H_1 \ 0] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \gamma,$$

(34a)

where

$$H = [U_1 S_1 \ 0] = [H_1 \ 0],$$

(34b)

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = V\tilde{r}_2.$$  

(34c)

It is clear that matrix $H_1 = U_1 S_1$ has full column rank; therefore, by letting $\eta_2 = 0$, a solution to (34) can be obtained as

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} H_1^+ \gamma \\ 0 \end{bmatrix}.$$  

(35)

Vector $\tilde{r}_2$ is thus derived directly from (34c).

4.2. Stability analysis

As mentioned above, if (25) or its equivalence (30) is solved exactly, one can reconstruct a dynamic feedback (8) for the $i$th local control station with the error defined in (13) approaching asymptotically to zero, given a stable eigenstructure of matrix $E_i$.

In the case that a solution to (30) is obtained approximately as described in the previous section, and the control law (7b) may not be realized completely. One would then pose a question as to whether the global closed-loop system remains stable as a result of such an approximate solution for each local control station. In the following, an answer to this question is obtained by analysing the stability of the global closed-loop system.

Let us define the following error matrices:

$$\Delta \tilde{M}_i = (L_i A - E_i L_i) \begin{bmatrix} 0 \\ I_{n-r_i} \end{bmatrix},$$

(36a)

and

$$\Delta \tilde{F}_i = (F_i - K_i L_i) \begin{bmatrix} 0 \\ I_{n-r_i} \end{bmatrix},$$

(36b)

where matrices $\Delta \tilde{M}_i$ and $\Delta \tilde{F}_i$ are due to the approximate solution to (18b) and (19b) respectively.

In view of (15) and (17), the following error matrices are obtained:

$$\Delta M_i = [0 \ - \Delta \tilde{M}_i] = G_i C_i - L_i A + E_i L_i,$$

(37)

and

$$\Delta F_i = [0 \ \Delta \tilde{F}_i] = F_i - (K_i L_i + W_i C_i).$$

(38)

Hence, the error dynamic equation (14) now becomes

$$\dot{e}(t) = E\dot{e}(t) + \Delta M_i x(t),$$

(39)

and the feedback control law (7b) becomes

$$u_i(t) = (F_i + \Delta F_i)x(t), \quad i = 1, 2, \ldots, N.$$  

(40)

Equation (40) and (39) can also be cast in a compact form as

$$u(t) = (F + \Delta F)x(t)$$

(41)

and

$$\dot{e}(t) = E\dot{e}(t) + \Delta M x(t),$$

(42)

where

$$\Delta F = \begin{bmatrix} \Delta F_1 \\ \vdots \\ \Delta F_N \end{bmatrix}, \quad E = \text{diag}(E_i) \in R^{N_p \times N_p},$$

(43)

$$i = 1, 2, \ldots, N, \quad \Delta M = \begin{bmatrix} \Delta M_1 \\ \vdots \\ \Delta M_N \end{bmatrix}.$$  

By substituting (41) into (1a) and using (42), the following closed-loop system is obtained for the augmented state $w(t) = [x(t) \ e(t)]^T$:

$$\dot{w}(t) = (J + \Delta J)w(t),$$

(44a)

where

$$J = \begin{bmatrix} A + BF & 0 \\ 0 & E \end{bmatrix}, \quad \Delta J = \begin{bmatrix} B \Delta F \\ \Delta M \end{bmatrix}.$$  

(44b)

The nominal part $\dot{w}(t) = Jw(t)$ is stable and its eigenvalues are the union of the previously designed eigenvalues of the global control system and those of the observers, which are chosen as desired. The eigenvalues of $J + \Delta J$ can then be considered as perturbations of the eigenvalues of $J$. As $\Delta J$ has been minimized in (31) and matrix $J$ has been obtained to satisfy the stability and performance requirements, system (44) will be stable if the perturbation $\Delta J$ is sufficiently small.

In fact, it has been shown by Khargonekar et al. (1990) that the asymptotically stability of the closed-loop system (44) can be assured, if $\Delta J$ satisfies the stability bound

$$\|\Delta J\| < \alpha,$$  

(45a)
where  
\[ \alpha = \frac{1}{\|(sI - J)^{-1}\|_{\infty}}, \]  
(45b)

\( I \) is the unity matrix and \( \| \cdot \|_{\infty} \) denotes the \( H_{\infty} \) norm.

**Remark 3:** Since matrix \( E \) can be freely assigned, it should be chosen such that its poles lie to the left of the dominant poles of the global controller.

**Remark 4:** A close study of the minimization problem (31) reveals that \( \delta(K_i) \) (and hence the error matrix \( \Delta J \)) can be made successively smaller and smaller by increasing gradually the order \( p_i \) of the observer. This is clear from the pseudo-inverse matrix theory. Since increasing the order of the observer has the effect of making matrix \( \beta \) become squarer, provided that \( \beta \) has full column rank, the error norm, \( \delta(K_i) \) will therefore always become smaller and smaller. Accordingly, the closed-loop global stability and performance will be achieved by successively increasing the order of the observer (8b).

Let us now summarize the proposed approach from a practical point of view. After having obtained a global feedback controller that satisfies a system performance index, the design process starts by partitioning the global state feedback control and choosing a stable observer matrix for each local control station. The lowest order \( (p_i = 1) \) is first assigned for each observer of the form (8b). The optimization problem (31) is then formulated and solved to determine all the decentralized controller matrices. From the obtained results for all local control stations, the parameter \( \alpha \) and the norm of \( \Delta J \) are computed and the stability condition (45a) is tested. If this condition is satisfied, then closed-loop stability is guaranteed, and the design process is accomplished. Otherwise the order of the controller is increased by one and the procedure is repeated to result, as discussed in remark 4, in a smaller norm of \( \Delta J \) and, therefore, to meet eventually the closed-loop stability condition. This forms the basis for the following design algorithm.

**Design Algorithm:**

**Step 1.** Design a suitable global state feedback controller \( F \) by using any existing state-feedback controller design method. Set \( j = 0 \).

**Step 2.** Set the order of the observer (8b) as \( p_i = 1 + j \). Proceed with steps 3–6 for \( i = 1, 2, \ldots, N \).

**Step 3.** Select stable matrix \( E_i \) in accordance with remark 3.

**Step 4.** Partition system (1) according to (12). Partition matrices \( F_i \) and \( L_i \) according to (20) and (21) respectively.

**Step 5.** Solve the optimization problem (31) for \( K_i \) and \( \bar{L}_i \). Derive \( \bar{l}_i \) from (27). Obtain matrix \( T_i \) and derive vector \( l \) from \( l = T_i^\top \bar{l} \). Obtain \( L_i \) from (21).

**Step 6.** Derive matrices \( G_i \) and \( W_i \) from (18a) and (19a) respectively.

**Step 7.** Compute the stability bound \( \alpha \) and \( \| \Delta J \| \). Check whether the closed-loop stability condition (45a) is satisfied. If yes, then stop. Else, set \( j \rightarrow j + 1 \) and go to step 2.

**Remark 5:** The design procedure outlined above yields a lowest possible low-order linear functional observer for each local control station. The global system performance is then dependent on three factors:

(i) robustness of the adopted standard state feedback controller \( F \);

(ii) the resulting observer orders \( p_i \);

(iii) the choice of the observer matrices \( E_i \). 

The following numerical example will illustrate these points.

5. **Numerical example**

Consider an unstable fifth-order global system consisting of two control stations, that is \( N = 2 \). The decentralized information structure of the form (4) for each control station consists of one local control input and two local outputs. Owing to the constraint of the decentralized information structure (5), only local information is used to generate control signals. The system is described by (3), where matrices \( A, B_i \) and \( C_i (i = 1, 2) \) are given by

\[
A = \begin{bmatrix}
-3 & 0 & -0.6 & 1.5 & -0.3 \\
-0.3 & -6 & 0 & 0.6 & 1.5 \\
-1.2 & 1.5 & -9 & 0.3 & -3 \\
-2.25 & -0.6 & -2.4 & 2 & 0 \\
-0.6 & 1.5 & -1.5 & 1.5 & 3.75
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0.5 \\
0 \\
1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 \\
-2 \\
0.3
\end{bmatrix},
\]

\[
C_1 = \begin{bmatrix}
1 & 0.2 & -0.3 & 1 & 2 \\
1 & 0 & 0 & 0 & -0.5
\end{bmatrix},
\]

\[
C_2 = \begin{bmatrix}
0.5 & 0 & 0.1 & 0.7 & 0.9 \\
0.6 & 0.4 & -0.4 & 0.5 & 0
\end{bmatrix}
\]
The open-loop eigenvalues are \( \lambda(A) = \{-9.4163, 4.5047, -2.2316, 0.8752, -5.9874\} \). As can be seen, this system is unstable because of the presence of two unstable poles. In the following, we design two decentralized dynamic output feedback controllers of the form (8) for the two control stations, where each station uses only its local information. The design algorithm presented in section 4 is now followed.

As the system is controllable, a global linear state feedback control law can be easily derived. Using the linear–quadratic technique with the state and control weighting matrices chosen as \( Q = 2I_3 \) and \( R = I_2 \) respectively, the following optimal controller can be obtained:

\[
J = \begin{bmatrix}
   F_1 \\
   F_2
\end{bmatrix}
\]

where

\[
F_1 = [-0.9320 \quad 0.8682 \quad -1.1046 \quad 1.8187 \quad 7.6367],
F_2 = [-1.0132 \quad 0.3520 \quad -1.1734 \quad 3.6866 \quad 4.8109].
\]

Application of the above state feedback controller gives the following closed-loop global eigenvalues:

\[
\lambda(A + BF) = \{-9.4163, -6.0007, -3.4665 \pm j1.0292, -4.5434\}.
\]

Note that the point that this example seeks to illustrate is not to design the global controller itself but rather to generate any previously designed set of global control signals using the proposed decentralized functional observers (8).

Let the stable matrices \( E_i \) be chosen as \( E_i = -4 \). As a result, the upper stability bound \( \alpha \), defined in (45b), can be easily computed. This gives \( \alpha = 1.4695 \). Thus, provided that the norm of the error matrix \( \Delta J \) is less than this value (i.e. \( \|\Delta J\| < 1.4695 \)), the closed-loop system under the decentralized controllers (8) will be stable.

Let us now apply the design algorithm and by restricting the gain of matrices \( K_i \) to be in the range of \(-10 \leq K_i \leq 10\), \( i = 1, 2 \), the following matrices are derived: for subsystem 1,

\[
K_1 = 10, \quad \delta(K_1) = 0.0599,
L_1 = [0.1991 \quad 0.0537 \quad -0.0590 \quad 0.0201 \quad 0.2154],
W_1 = [1.6118 \quad -4.5317],
G_1 = [0.7590 \quad -0.6833],
\]

and for subsystem 2,

\[
K_2 = 10, \quad \delta(K_2) = 1.0674,
W_2 = [5.0508 \quad -4.7868],
G_2 = [2.2560 \quad -1.8216],
L_2 = [-0.0418 \quad 0.2213 \quad -0.3469 \quad 0.2387 \quad 0.0235].
\]

Using the above data, the error matrix \( \Delta J \) and its norm can be computed. This gives \( \|\Delta J\| = 1.0833 \). As the stability condition (45a) is satisfied, the following two first-order decentralized dynamical output feedback controllers (8) for the two control stations are obtained for control station 1: the first-order decentralized output feedback controller is

\[
u_1(t) = 10z_1(t) + [1.6118 \quad -4.5317]y_1(t),
\]

\[
\dot{z}_1(t) = -4z_1(t) - 0.0257u_1(t) + [0.7590 \quad -0.6833]y_1(t).
\]

and for control station 2: the first-order decentralized output feedback controller is

\[
u_2(t) = 10z_2(t) + [5.0508 \quad -4.7868]y_2(t),
\]

\[
\dot{z}_2(t) = -4z_2(t) - 0.8478u_2(t) + [2.2560 \quad -1.8216]y_2(t).
\]

The above two first-order decentralized controllers stabilize the system with the closed-loop eigenvalues are as shown in table 1 together with the norm of the error matrix, \( \Delta J \), and the two error terms \( \delta(K_i) \)

<table>
<thead>
<tr>
<th>First-order controllers</th>
<th>Second-order controllers</th>
<th>Third-order controllers</th>
<th>Global full state feedback controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 1.4695 )</td>
<td>( \alpha = 1.4695 )</td>
<td>( \alpha = 1.4695 )</td>
<td>( \alpha = 1.4695 )</td>
</tr>
<tr>
<td>( |\Delta J| = 1.0833 )</td>
<td>( |\Delta J| = 0.2014 )</td>
<td>( |\Delta J| = 0 )</td>
<td>( |\Delta J| = 0 )</td>
</tr>
<tr>
<td>( \delta(K_1) = 0.0599 )</td>
<td>( \delta(K_1) = 0.0344 )</td>
<td>( \delta(K_1) = 0 )</td>
<td>( \delta(K_1) = 0 )</td>
</tr>
<tr>
<td>( \delta(K_2) = 1.0674 )</td>
<td>( \delta(K_2) = 0.1992 )</td>
<td>( \delta(K_2) = 0 )</td>
<td>( \delta(K_2) = 0 )</td>
</tr>
<tr>
<td>(-8.9364 \pm j2.5146 )</td>
<td>(-6.5978 )</td>
<td>(-9.4163 )</td>
<td>(-9.4163 )</td>
</tr>
<tr>
<td>(-6.1803 )</td>
<td>(-4.4211 \pm j2.0569 )</td>
<td>(-6.0007 )</td>
<td>(-6.0007 )</td>
</tr>
<tr>
<td>(-3.7520 )</td>
<td>(-2.8229 )</td>
<td>(-3.4665 \pm j1.0292 )</td>
<td>(-3.4665 \pm j1.0292 )</td>
</tr>
<tr>
<td>(-4.6058 )</td>
<td>(-3.7320 )</td>
<td>(-4.5434 )</td>
<td>(-4.5434 )</td>
</tr>
<tr>
<td>(-0.9963 \pm j1.4939 )</td>
<td>(-5.0075 )</td>
<td>(-4, -5, -6 )</td>
<td>(-4, -5, -6 )</td>
</tr>
<tr>
<td>(-4.2956 )</td>
<td>(-3.6063 )</td>
<td>(-6 )</td>
<td>(-6 )</td>
</tr>
</tbody>
</table>
decentralized output feedback controller is derived: for control station 1 the second-order decentralized output feedback controllers for the two control stations are derived: for control station 1 the second-order decentralized output feedback controller is

\[
  u_1(t) = [100] z_1(t) + [1.6188 -4.9689] y_1(t),
\]

\[
  \dot{z}_1(t) = \text{diag}[-4, -5] z_1(t) + \begin{bmatrix} -0.0016 \\ 0.0411 \\ 0.0746 \\ -0.0656 \\ 0.7333 \\ -0.3853 \end{bmatrix} y_1(t),
\]

and for control station 2 the second-order decentralized output feedback controller is

\[
  u_2(t) = [100] z_2(t) + [6.4711 -5.5357] y_2(t),
\]

\[
  \dot{z}_2(t) = \text{diag}[-4, -5] z_2(t) + \begin{bmatrix} 1.6150 \\ -2.477 \\ -12.7758 \\ 8.1054 \\ 14.938 \\ -7.8265 \end{bmatrix} y_2(t).
\]

For comparison purposes, Table 1 also shows the error terms \( \|\Delta J\|, \delta(K_i) \) \( i = 1, 2 \) and the distribution of eigenvalues of the closed-loop global system under the two second-order decentralized controllers. It is clear from Table 1 that the error terms decrease because of an increase in the observer order (from 1 to 2). In addition, the dominant eigenvalues move closer to the dominant eigenvalues of the global controller. Naturally, it is therefore expected that the performance of the closed-loop system under the two second-order decentralized controllers is better than that of the two first-order decentralized controllers.

Finally, for completeness, the design algorithm is repeated for the case where the order of the observers is increased from 2 to 3, as follows: for control station 1, the third-order decentralized output feedback controller is

\[
  u_1(t) = [111] z_1(t) + [1.6179 -5.2988] y_1(t),
\]

\[
  \dot{z}_1(t) = \text{diag}[-4, -5, -6] z_1(t) + \begin{bmatrix} 0.1630 \\ 0.7064 \\ 0.0192 \end{bmatrix} y_1(t),
\]

\[
  + \begin{bmatrix} -3.6285 \\ 3.3467 \\ 11.7300 \\ -5.8796 \\ 0.2980 \\ -0.3523 \end{bmatrix} y_1(t),
\]

and, for control station 2, the third-order decentralized output feedback controller is

\[
  u_2(t) = [100] z_2(t) + [6.6647 -5.6187] y_2(t),
\]

\[
  \dot{z}_2(t) = \text{diag}[-4, -5, -6] z_2(t) + \begin{bmatrix} 1.4617 \\ -2.0108 \\ -0.3069 \\ -11.5708 \\ 7.3891 \\ 11.5797 \\ -6.0842 \\ 1.4570 \\ -0.7892 \end{bmatrix} y_2(t).
\]

In this case, the error terms \( \|\Delta J\|, \delta(K_i) \) \( i = 1, 2 \) vanish, as shown in Table 1. This implies that (30) and (25) is solved exactly and the two third-order decentralized controllers emulate completely the global state feedback controller. This result coincides with remark 1, which suggests the choice of the observer order for an exact solution to be \( p_i \geq m_i(n - r_i)/(r_i + m_i - m_i) = (5 - 2)/(2 + 1 - 2) = 3, \) \( i = 1, 2 \). The closed-loop eigenvalues of the three controllers as well as of the global state controller are also shown in Table 1, for a comprehensive comparison.

Note that, for the third-order controllers, the closed-loop eigenvalues are the union of the eigenvalues of the global state feedback controller and of the two decentralized observers, which is in line with the separation principle.

6. Conclusion

This paper has considered the problem of controlling a multi-agent system under an N-decentralized information structure. An approach to decentralized control implementation of a global controller is proposed using reduced-order linear functional observers. It is shown that, if a solution for the observer matrices is not obtained exactly, the proposed decentralization of the global control task can still be implemented approximately following the minimization of a parameter function. Stability analysis for the closed-loop system is included. A step-by-step design algorithm is presented for the design of N decentralized observers. Features of the proposed design method are illustrated through a numerical example.

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References


