# On Input to State Stabilization for Nonlinear Discrete-Time Systems: a Dynamic Programming Approach*

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**Abstract**

A novel technique for synthesizing dynamic state feedback controllers that achieve input to state stability (ISS) property for nonlinear discrete-time systems is presented. The ISS controller design for the original system is solved via an auxiliary $l^\infty$-bounded (LIB) robustness synthesis problem for an auxiliary system. A complete solution for the LIB synthesis problem has been presented recently in the literature. The obtained static state feedback LIB controller for the auxiliary system can be interpreted as a dynamic state feedback ISS controller for the original system. The obtained solution is in terms of a dynamic programming equation (or inequality).

**Keywords:** Input to state stability (ISS), nonlinear robust control, $l^\infty$ criteria, dynamic programming, controller synthesis.

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**Preference:** presented in a lecture session.

## 1 Introduction

The *input to state stability* (ISS) property for systems with disturbances was first proposed by Sontag in 1989 [11]. Since then, ISS has received a lot of attention with a

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range of its applications and different characterizations reported in the literature. For example, a systematic analysis of the ISS property has been conducted in [14, 15], where its many different characterizations have been described. The discrete-time ISS property and ISS small-gain theorems were studied in [9]. Further results on ISS and many related properties can be found in the survey paper [13] and the references therein.

ISS can be regarded as a particular type of $L^\infty$ (or $l^\infty$) stability that is fully compatible with Lyapunov theory and in particular it can be checked using the so called ISS Lyapunov functions. To date there is no systematic way to generate ISS Lyapunov functions. Besides ISS, a range of alternative $L^\infty$ (or $l^\infty$) stability properties have been investigated recently in [2, 3, 7]. An interesting difference between this literature and the ISS related literature is that the analysis is carried out via robust optimal control techniques instead of Lyapunov based methods. These optimization approaches typically make use of appropriate dynamic programming equations that solve the analysis and synthesis problem. It appears that investigation of the ISS property via optimization based techniques such as dynamic programming is an open question in the literature.

In this paper, we consider the ISS controller synthesis problem when the disturbance gains and bounds on transients are given. A related analysis problem of finding the "minimal" ISS gain and transient bounds is considered in [6]. For simplicity, we present only results on full state feedback control – the partial state feedback (measurement) problem is solved in a forthcoming paper. We present a new technique for the synthesis of controllers achieving ISS that is based on a recently obtained result on $l^\infty$-bounded (LIB) robustness for nonlinear systems [5]. By introducing two new state variables, the ISS controller synthesis problem for the original system is transformed into an equivalent uniform LIB dissipation synthesis problem for an auxiliary system. The full state feedback solution for the LIB problem for the auxiliary system is then interpreted as a dynamic full state feedback ISS controller for the original system. Dynamic programming techniques are used to obtain necessary and sufficient conditions for the existence of such controllers that yield ISS for the closed loop system with a given disturbance gain and transient bound.

The paper is organized as follows. Preliminaries are presented in Section 2. In Section 3, the state feedback ISS synthesis problem and the state feedback uniform LIB synthesis problem are stated. In Section 4, we transformed the ISS synthesis problem into a uniform LIB synthesis problem for an auxiliary system and use uniform LIB results to obtain a solution to ISS synthesis problem. Two examples are given in Section 5 to illustrate the method. The related results concerning uniform LIB dissipation are provided in the Appendix.

## 2 Preliminaries

Sets of real numbers, nonnegative real numbers, integers and nonnegative integers are denoted respectively as $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$ and $\mathbf{Z}_+$. Moreover, we denote $\bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$. Recall that a function $\gamma : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also $\gamma(s) \to \infty$ as $s \to \infty$. A function

$\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be a function of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$ $\beta(s,\cdot)$ decreases to zero.

Given $w_k \in \mathbf{W} \subseteq \mathbf{R}^s, \forall k \in \mathbf{Z}_+$, we exploit the following notation:

$$\begin{aligned} w_{0,k-1} &= \{w_0, \cdots, w_{k-1}\}, \forall k \geq 0, \\ \mathcal{W}_{0,k-1} &= \{w_{0,k-1} : w_i \in \mathbf{W}, 0 \leq i \leq k-1\}, \\ \mathcal{W}_{0,\infty} &= \{w_{0,\infty} : w_i \in \mathbf{W}\}. \end{aligned} \tag{1}$$

In the sequel, $x_{0,k}, \mathcal{X}_{0,k}, \mathcal{X}_{0,\infty}, u_{0,k}, \mathcal{U}_{0,k}, \mathcal{U}_{0,\infty}$ have the similar meaning for $x_k \in \mathbf{R}^n$ and $u_k \in \mathbf{R}^m$. We also use the following notation:

$$\|w_{0,\infty}\|_\infty = \sup_{i\geq 0} |w_i| ; \qquad \|w_{0,k-1}\|_\infty = \max_{0\leq i\leq k-1} |w_i|$$

where $|\cdot|$ is the Euclidean norm.

**Definition 2.1** *A map $K : \mathcal{X}_{0,\infty} \to \mathcal{U}_{0,\infty}$ is* causal *if its value at any time $k$ is independent of $\mathcal{X}_{k+1,\infty}$ meaning that for each time $k \geq 0$ if $x^1, x^2 \in \mathcal{X}_{0,\infty}$ and $x^1_l = x^2_l$ for all $0 \leq l \leq k$ then $K(x^1)_k = K(x^2)_k$.*

The following lemma is used in the sequel.

**Lemma 2.2** *[12, Proposition 7] Given any $\beta \in \mathcal{KL}$, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\beta(s,t) \leq \alpha_1(\alpha_2(s)e^{-t}), \ \forall s,t \geq 0. \tag{2}$$

Hence, there is no loss of generality to suppose that $\beta \in \mathcal{KL}$ has the form

$$\beta(s,k) = \alpha_1(\alpha_2(s)e^{-k}), \ \forall s \geq 0, \ \forall k \in Z_+. \tag{3}$$

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

# 3 The ISS and LIB Controller Synthesis Problems

In this section we define two problems that we investigate in the sequel. Consider the nonlinear discrete-time system

$$x_{k+1} = f(x_k, u_k, w_k), \ k \geq 0 \tag{4}$$

Here $x_k \in \mathbf{R}^n, u_k \in \mathbf{U} \subseteq \mathbf{R}^m$ and $w_k \in \mathbf{W} \subseteq \mathbf{R}^s$ are the state, control input and input disturbance, respectively, $f : \mathbf{R}^n \times \mathbf{U} \times \mathbf{W} \to \mathbf{R}^n$.

The class of admissible controllers for the plant (4) that we consider is defined below.

**Definition 3.1** *An* admissible *state feedback controller is a causal map $K : \mathcal{X}_{0,\infty} \to \mathcal{U}_{0,\infty}$. The set of all admissible state feedback controllers is denoted as $\mathcal{K}_{state}$.*

We sometimes abuse the notation by writing $u_k = K(x_{0,k})$ or $u = K(x)$.

The problem that we consider in this paper is stated next.

> **State Feedback ISS Synthesis Problem (SFISS):** Given $\gamma \in \mathcal{K}$, $\omega_1 : \mathbf{R}^n \to \mathbf{R}$, $\omega_2 : \mathbf{R}^n \to \mathbf{R}$, $\lambda \in \mathbf{R}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ that define $\beta \in \mathcal{KL}$ via (3), find an admissible state feedback controller $K \in \mathcal{K}_{state}$ such that the trajectories of the closed-loop system consisting of the controller $K(\cdot)$ and the plant (4) satisfy
>
> $$|\omega_1(x_k)| \le \beta(|\omega_2(x_0)|, k) + \gamma(\|w_{0,\infty}\|_\infty) + \lambda, \tag{5}$$
>
> for all $x_0 \in \mathbf{R}^n$, $w_{0,\infty} \in \mathcal{W}_{0,\infty}$ and $k \in \mathbf{Z}_+$.

When the trajectories of the closed-loop system satisfy the above bound, then we say that the closed-loop system is ISS. Note that from causality and the form of (3), the inequality (5) is equivalent to the following inequality

$$|\omega_1(x_k)| \le \ \alpha_1(\alpha_2(|\omega_2(x_0)|)e^{-k}) + \gamma(\|w_{0,k-1}\|_\infty) + \lambda \tag{6}$$

for all $x_0 \in \mathbf{R}^n$, $w_{0,k-1} \in \mathcal{W}_{0,k-1}$ and $k \in \mathbf{Z}_+$.

Inequality (5) is a compact way to write a range of ISS like properties that have been considered in the literature. Indeed, different forms of functions $\omega_1, \omega_2$, different values for $\lambda$, and different sets $\mathbf{W}$ result in different kind of properties that have been considered in the literature. For example, if $\gamma(s) \equiv 0$ and $\lambda = 0$ we obtain the stability with respect to two measures considered in [10]. If we let $\omega_1(x) = \omega_2(x) = x$, $\lambda = 0$, $\mathbf{W} = \mathbf{R}^s$, we obtain the standard Input to State Stability (ISS) property [11, 9]. When $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}, \mathbf{W} = \mathbf{R}^s$, the property is the Input to State practical Stability (ISpS) with respect to set $\mathcal{A}$ that is not necessarily compact [8, 15]. When $\lambda = 0, \mathbf{W} = \mathbf{R}^s$, and $\omega_1(x) = h(x), \omega_2(x) = x$ where $h(x)$ defines the output function, i.e. $y_k = h(x_k)$, then the property is the Input to Output Stability (IOS) property [16]. When $\mathbf{W} \subsetneqq \mathbf{R}^s$, the corresponding properties are local ISS-like properties.

We will show that the SFISS problem for system (4) can be solved by solving the following controller synthesis problem for an auxiliary system. We first state the problem itself and then introduce the auxiliary system in the next section. To state the following problem we also need to introduce the performance output equation for system (4)

$$z_k = g(x_k), \ k \ge 0 \tag{7}$$

where $z_k \in \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}$.

> **State Feedback Uniform LIB Synthesis Problem (SFULIB):** Given $B_0 \subseteq \mathbf{R}^n$ and $\lambda \in \mathbf{R}$, find an admissible state feedback controller $K \in \mathcal{K}_{state}$ such that the trajectories of the closed-loop system consisting of the plant (4), (7) and the controller $K(\cdot)$ satisfy
>
> $$z_k \le \lambda, \tag{8}$$
>
> for all $x_0 \in B_0$, $w_{0,k-1} \in \mathcal{W}_{0,k-1}$ and $k \ge 0$.

When the trajectories of the closed-loop system satisfy the above bound, we say that the closed-loop system is uniform $l^\infty$-bounded (LIB) dissipative with respect to $B_0$ and $\lambda$.

# 4 Solution to the ISS Synthesis Problem

In this section we show how we can solve the SFISS problem for system (4) by solving the SFULIB problem for an auxiliary system that is constructed next by augmentation of the state variables, appropriately defined performance output equation and the set $\bar{B}_0$ on which the uniform LIB property should hold. We emphasize that the solution to the SFULIB problem has been already obtained in [5] and it is summarized in the Appendix A for completeness.

To this end, suppose $x_0, u_{0,k-1}$ are fixed, and some $w_{0,k-1}$ result in the same $x_{1,k}$. We will be most interested in the $w_{0,k-1}$ such that $\|w_{0,k-1}\|_\infty$ is the smallest. Since if (6) holds for this $w_{0,k-1}$, then it will also holds for the other $w_{0,k-1}$. This motivates us to define the following function $\hat{f}(x_0, u_0, x_1)$. For $x_0, x_1 \in \mathbf{R}^n, u_0 \in \mathbf{U} \subseteq \mathbf{R}^m$ such that $f(x_0, u_0, w_0) = x_1$ for some $w_0 \in \mathbf{W}$, we denote

$$\hat{f}(x_0, u_0, x_1) = \min_{w \in \mathbf{W}} \{|w| : f(x_0, u_0, w) = x_1\}. \tag{9}$$

Notice that $\hat{f}(x_0, u_0, x_1)$ is well defined and $0 \leq \hat{f}(x_0, u_0, x_1) \leq |w_0|$.

Consider system (4) and let $\alpha_1, \alpha_2, \gamma, \omega_1, \omega_2, \lambda$ come from the inequality (6). The auxiliary system is defined as follows:

$$\begin{aligned} \xi_{k+1} &= \tilde{f}(\xi_k, u_k, w_k), \ k \geq 0 \\ z_k &= g(\xi_k), \ k \geq 0 \end{aligned} \tag{10}$$

where

$$\xi_k = \begin{pmatrix} x_k \\ \zeta_k \\ \eta_k \end{pmatrix}, \tag{11}$$

$$\tilde{f}(\xi_k, u_k, w_k) = \begin{pmatrix} f(x_k, u_k, w_k) \\ e^{-1}\zeta_k \\ \max\{\eta_k, \hat{f}(x_k, u_k, f(x_k, u_k, w_k))\} \end{pmatrix}, \tag{12}$$

$$g(\xi_k) = |\omega_1(x_k)| - \alpha_1(\zeta_k) - \gamma(\eta_k). \tag{13}$$

Also we define

$$\bar{B}_0 := \left\{ \begin{pmatrix} x_0 \\ \alpha_2(|\omega_2(x_0)|) \\ 0 \end{pmatrix} : x_0 \in \mathbf{R}^n \right\} \subseteq \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+ . \tag{14}$$

Since the system (10) is higher dimensional than (4), we find it convenient to introduce different notation for sets of admissible controllers. The set of admissible controllers for (10) and (4) are respectively denoted as $\bar{\mathcal{K}}_{state}$ and $\mathcal{K}_{state}$.

**Lemma 4.1** *The SFISS problem for system (4) with given* $\alpha_1, \alpha_2, \gamma, \omega_1, \omega_2, \lambda$ *is equivalent to the SFULIB problem for system defined by (10)-(13) with* $\bar{B}_0$ *defined in (14) and* $\lambda$*. That is, the SFISS problem for system (4) with given* $\alpha_1, \alpha_2, \gamma, \omega_1, \omega_2, \lambda$ *has a solution* $K \in \mathcal{K}_{state}$ *if and only if the SFULIB problem for the system defined by (10)-(13) with* $\bar{B}_0$ *and* $\lambda$ *has a solution* $\bar{K} \in \bar{\mathcal{K}}_{state}$*.*

Proof. Suppose the SFISS problem for system (4) has a solution $K \in \mathcal{K}_{state}$, then $K \in \bar{\mathcal{K}}_{state}$ because $\mathcal{K}_{state} \subseteq \bar{\mathcal{K}}_{state}$. Consider the closed-loop system combining (10) with controller $K$.

For $\xi_0 = (x_0, \zeta_0, \eta_0) \in \bar{B}_0$ and $w_{0,k-1} \in \mathcal{W}_{0,k-1}$, from

$$x_0 \in \mathbf{R}^n, \zeta_0 = \alpha_2(|\omega_2(x_0)|), \eta_0 = 0,$$

$$\zeta_{i+1} = e^{-1}\zeta_i, \quad \eta_{i+1} = \max\{\eta_i, \hat{f}(x_i, u_i, f(x_i, u_i, w_i))\}, \ i \geq 0$$

and the definition of $\hat{f}$ (in (9)), we have

$$\zeta_k = \alpha_2(|\omega_2(x_0)|)e^{-k}, \quad \eta_k = \min\{\|\hat{w}_{0,k-1}\|_\infty : f(x_i, u_i, \hat{w}_i) = f(x_i, u_i, w_i), 0 \leq i \leq k-1\}.$$

Hence inequality (6) implies

$$z_k = g(\xi_k) = |\omega_1(x_k)| - \alpha_1(\zeta_k) - \gamma(\eta_k) \leq \lambda. \tag{15}$$

So the controller $K$ solves the SFULIB problem for system defined by (10)-(13) with $\bar{B}_0$ and $\lambda$.

Conversely, suppose the SFULIB problem for system defined by (10)-(13) with $\bar{B}_0$ and $\lambda$ has a solution $\bar{K} \in \bar{\mathcal{K}}_{state}$. Since $\zeta_0 = \alpha_2(|\omega_2(x_0)|), \eta_0 = 0$ and $\zeta_k, \eta_k$ $(k \geq 1)$ can be obtained from $x_k$ $(k \geq 0)$ by

$$\begin{aligned} \zeta_{k+1} &= e^{-1}\zeta_k, \ k \geq 0 \\ \eta_{k+1} &= \max\{\eta_k, \hat{f}(x_k, u_k, x_{k+1})\}, \ k \geq 0, \end{aligned} \tag{16}$$

we can use $\bar{K}$ to obtain a controller $K \in \mathcal{K}_{state}$. It is easy to show that the closed-loop system combining (4) with this controller $K$ is ISS using similar argument as above. □

Using Lemma 4.1 and the results of state feedback uniform LIB synthesis (see Appendix A), we have the following theorems.

**Theorem 4.2** *(Necessity) If there exists a state feedback controller $K_0 \in \mathcal{K}_{state}$ such that the closed-loop system (consisting (4) with $K = K_0$) is ISS, then the function $V_a : \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+ \to \bar{\mathbf{R}}$ defined by*

$$V_a(\xi) = \inf_{\bar{K} \in \bar{\mathcal{K}}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \{g(\xi_k) : u = \bar{K}(\xi), \xi_0 = \xi\}, \ \forall \xi \in \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+ \tag{17}$$

*satisfies:*

1. *$\bar{B}_0 \subseteq \mathrm{dom}V_a \overset{\triangle}{=} \{\xi \in \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+ : V_a(\xi) < +\infty\}$, $\sup_{\xi \in \bar{B}_0} V_a(\xi) \leq \lambda$;*

2. *$V_a$ solves the dynamic programming equation (DPE)*

$$V_a(\xi) = \max\{g(\xi), \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(\tilde{f}(\xi, u, w))\}, \ \forall \xi \in \mathrm{dom}V_a. \tag{18}$$

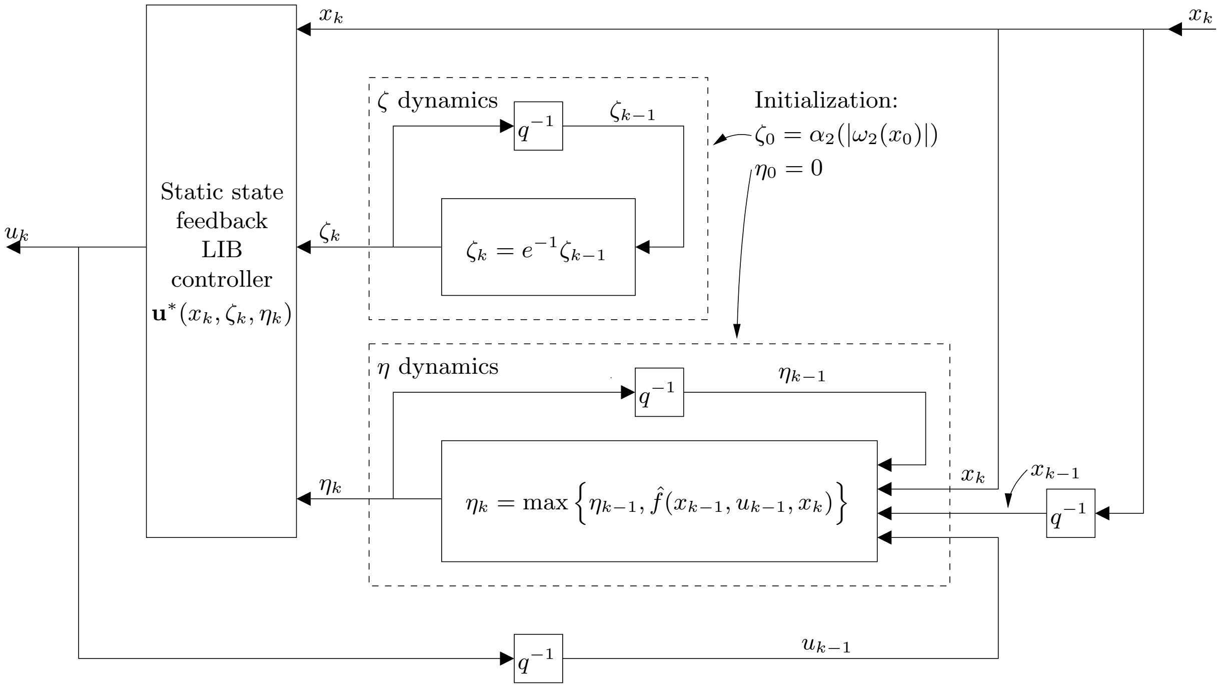

Figure 1: The dynamic state feedback controller $K_{ISS}$, where $q^{-1}$ denotes the a step delay.

Let $S \subseteq \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+$ and $V : \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+ \to \bar{\mathbf{R}}$. Let $(V, S)$ solve the dynamic programming inequality (DPI)

$$V(\xi) \geq \max\{g(\xi), \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V(\tilde{f}(\xi, u, w))\}, \ \forall \xi \in S, \tag{19}$$

$(V, S)$ is said to be a *good* solution to (19) if the infimum in (19) is attained by a function $\mathbf{u}^*(\xi)$ for all $\xi \in S$ and $S$ is invariant under the closed-loop dynamics determined by $\mathbf{u}^*$. See the Appendix A for further information.

**Theorem 4.3** *(Sufficiency) If $(V, S)$ is a good solution to the DPI (19) with $\bar{B}_0 \subseteq S$ and $\sup_{\xi \in \bar{B}_0} V(\xi) \leq \lambda$, then we can use the static state feedback controller $u_k = \mathbf{u}^*(\xi_k)$ to construct a dynamic state feedback controller $K_{ISS} \in \mathcal{K}_{state}$ by*

$$\begin{cases} \zeta_{k+1} &= e^{-1}\zeta_k, \ k \geq 0 \\ \eta_{k+1} &= \max\{\eta_k, \hat{f}(x_k, u_k, x_{k+1})\}, \ k \geq 0 \\ u_k &= \mathbf{u}^*(\xi_k) = \mathbf{u}^*(x_k, \zeta_k, \eta_k), \ k \geq 0 \end{cases} \tag{20}$$

*where $\zeta_0 = \alpha_2(|\omega_2(x_0)|), \eta_0 = 0$ and $x_k \in \mathbf{R}^n, k \geq 0$ are known. The closed-loop system combining (4) with $K_{ISS}$ is ISS.*

The structure of the dynamic state feedback controller $K_{ISS}$ is shown in Figure 1.

**Remark 4.4** The variables $\zeta_k$ and $\eta_k$ in the dynamic controller satisfies

$$\begin{aligned} |\zeta_k| &\leq e^{-k}|\zeta_0|, \\ |\eta_k| &\leq ||w_{0,\infty}||_\infty, (\text{when } \eta_0 = 0) \end{aligned} \tag{21}$$

which are also ISS inequalities.

**Remark 4.5** The ISS property is qualitatively equivalent to the stability definition when $\beta(|\omega_2(x_0)|, k) + \gamma(\|w_{0,\infty}\|_\infty)$ is replaced by $\max\{\beta(|\omega_2(x_0)|, k), \gamma(\|w_{0,\infty}\|_\infty)\}$ in (5) (see e.g. [9, equation (21)]). We remark here that we can also deal with the $\max\{\beta, \gamma\}$ case by simply changing the $g(\xi)$ function in (13). Moreover, the integral ISS problems [1] can also be dealt with using similar methods.

# 5 Examples

Generally speaking, it is not possible to obtain explicit formulas for solutions $(V, S)$ to the DPI (19) or DPE (18). However, in some special cases, the computation can be simplified significantly, making it possible to obtain an explicit solution; this is done in section 5.1. In the following section 5.2 we look at a more complicated example numerically.

## 5.1 An Example with Explicit Solution

Consider one-dimensional discrete-time system with dynamics:

$$x_{k+1} = f(x_k, u_k) + w_k \tag{22}$$

where $x_k, w_k, u_k \in \mathbf{R}$ and function $f$ satisfies

$$\forall x \in \mathbf{R}, \exists \mathbf{u}^*(x) \in \mathbf{R}, \text{ such that } f(x, \mathbf{u}^*(x)) = 0. \tag{23}$$

Consider the SFISS problem with $\omega_1(x) = \omega_2(x) = x$. Suppose

$$\beta(s, k) = se^{-k}, \quad \gamma(\delta) = \delta, \quad \lambda = 0. \tag{24}$$

i.e. $\alpha_1(s) = \alpha_2(s) = s$.

For this example,

$$\hat{f}(x_0, u, x_1) = |x_1 - f(x_0, u)|. \tag{25}$$

Hence

$$\begin{aligned} \tilde{f}(\xi_k, u_k, w_k) &= \begin{pmatrix} f(x_k, u_k, w_k) \\ e^{-1}\zeta_k \\ \max\{\eta_k, |w_k|\} \end{pmatrix}, \\ g(\xi_k) &= |x_k| - \zeta_k - \gamma(\eta_k). \end{aligned} \tag{26}$$

We first solve the corresponding SFULIB problem. The value function is

$$V_a(\xi) \overset{\triangle}{=} \inf_{\bar{K} \in \bar{\mathcal{K}}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1}} \{g(\xi_k) : \xi_0 = \xi, u = K(\xi)\}. \tag{27}$$

For fixed $\xi_0 = (x_0, \zeta_0, \eta_0)$,

$$g(\xi_0) = |x_0| - \zeta_0 - \eta_0. \tag{28}$$

For $u_0$ and $w_0$,

$$\begin{aligned} g(\xi_1) &= |x_1| - \zeta_1 - \eta_1 \\ &= |f(x_0,u_0) + w_0| - e^{-1}\zeta_0 - \max\{\eta_0, |w_0|\}. \end{aligned} \tag{29}$$

For fixed $u_0$,

$$\begin{aligned} \sup_{w_0} g(\xi_1) &= \sup_{w_0}\{|f(x_0,u_0) + w_0| - e^{-1}\zeta_0 - \max\{\eta_0, |w_0|\}\} \\ &= -e^{-1}\zeta_0 + \sup_{w_0}\{|f(x_0,u_0) + w_0| - \max\{\eta_0, |w_0|\}\} \\ &= -e^{-1}\zeta_0 + |f(x_0,u_0)|. \end{aligned} \tag{30}$$

Similarly, for fixed $u_0, u_1$, we have

$$\sup_{w_1} g(\xi_2) = -e^{-1}\zeta_1 + |f(x_1,u_1)| = -e^{-2}\zeta_0 + |f(x_1,u_1)|. \tag{31}$$

Repeat the above process, for $u_0, u_1, \cdots$, we have

$$\begin{aligned} &\sup_{k\geq 0} \sup_{w_{0,k-1}} \{g(\xi_k)\} \\ = &\max\left\{ g(\xi_0), -e^{-1}\zeta_0 + |f(x_0,u_0)|, \sup_{k\geq 1} \sup_{w_{0,k-1}} \{-e^{-(k+1)}\zeta_0 + |f(x_k,u_k)|\} \right\}. \end{aligned} \tag{32}$$

Now we obtain the value function

$$\begin{aligned} V_a(x,\zeta,\eta) &= V_a(\xi) \\ &= \inf_{\bar{K}\in\bar{\mathcal{K}}_{state}} \sup_{k\geq 0} \sup_{w_{0,k-1}} \{g(\xi_k) : \xi_0 = \xi, u = K(\xi)\} \\ &= \max\{g(\xi), 0\} \\ &= \max\{|x| - \zeta - \eta, 0\}. \end{aligned} \tag{33}$$

By (26), we have

$$\begin{aligned} \inf_u \sup_w V_a(\tilde{f}(\xi,u,w)) &= \inf_u \sup_w \max\{|f(x,u) + w)| - e^{-1}\zeta - \max\{\eta, |w|\}, 0\}\} \\ &= \inf_u \max\{-e^{-1}\zeta + |f(x,u)|, 0\}. \end{aligned} \tag{34}$$

Hence

$$\inf_u \sup_w V_a(\tilde{f}(\xi,u,w)) = \sup_w V_a(\tilde{f}(\xi, \mathbf{u}^*(\xi), w)) = \max\{-e^{-1}\zeta, 0\} = 0 \tag{35}$$

where

$$\mathbf{u}^*(\xi) = \mathbf{u}^*(x). \tag{36}$$

Obviously

$$\max\{g(\xi), \inf_u \sup_w V_a(\tilde{f}(\xi,u,w))\} = \max\{|x| - \zeta - \eta, 0\} = V_a(\xi), \quad \forall \xi \in \mathrm{dom} V_a. \tag{37}$$

i.e. the DPE holds.

The controller $K_{ISS}$ in (20) is

$$u_k = \mathbf{u}^*(x_k). \tag{38}$$

When applying the above controller, the closed-loop system becomes

$$x_{k+1} = w_k. \tag{39}$$

Obviously, it is ISS with the $\beta, \gamma, \lambda$ defined by (24).

## 5.2 An Example with Numerical Solution

Consider system

$$x_{k+1} = x_k^3 + (x_k^2 + 1)u_k + \frac{1}{1 + x_k^2 + u_k^2} w_k. \tag{40}$$

We consider the SFISS problem for the $\beta, \gamma, \lambda$ defined by (24). We use a numerical algorithm to solve the DPE

$$\begin{aligned} V_a(x, \zeta, \eta) &= V_a(\xi) \\ &= \max\{g(\xi), \inf_u \sup_w V_a(\tilde{f}(\xi, u, w)) \\ &= \max\{|x| - \zeta - \eta, \inf_u \sup_w V_a(\tilde{f}(\xi, u, w))\}. \end{aligned} \tag{41}$$

The obtained value function $V_a(x, \zeta, \eta)$ together with the optimal controller $\mathbf{u}^*(x, \zeta, \eta)$ are given in Figure 2.

The controller $K_{ISS}$ in (20) is given in (a) of Figure 3. A simulation of the closed-loop system is illustrated in (b) of Figure 3, which demonstrates consistency with the ISS inequality.

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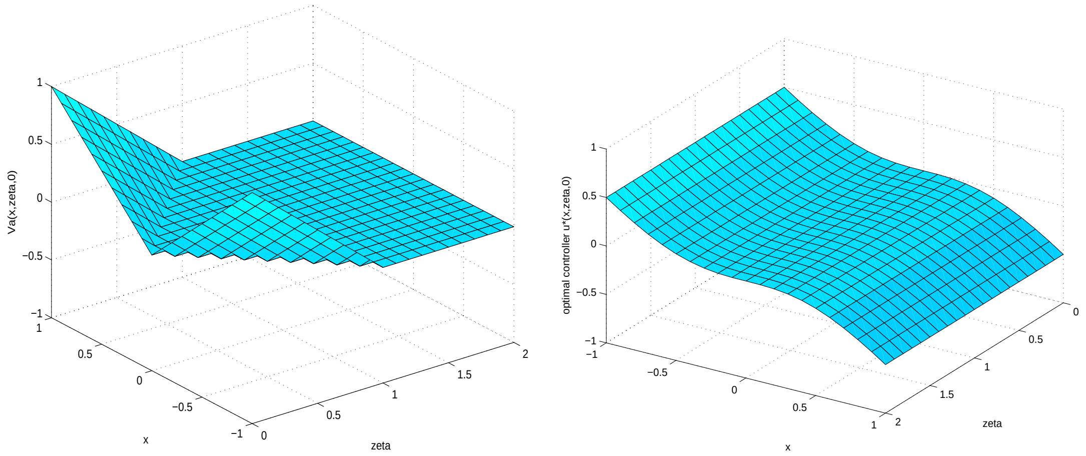
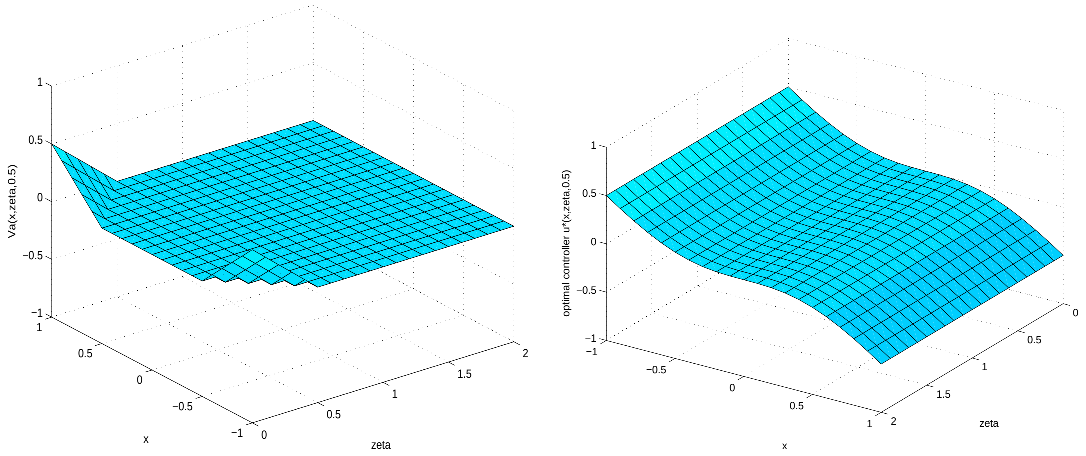
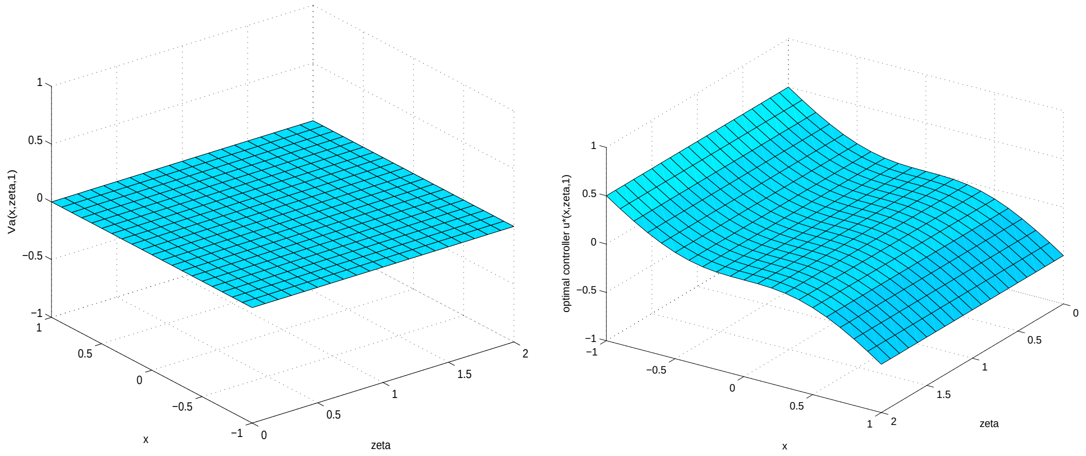


(a) Value function $V_a$: $\eta = 0$ (b) optimal controller $u^*$: $\eta = 0$.

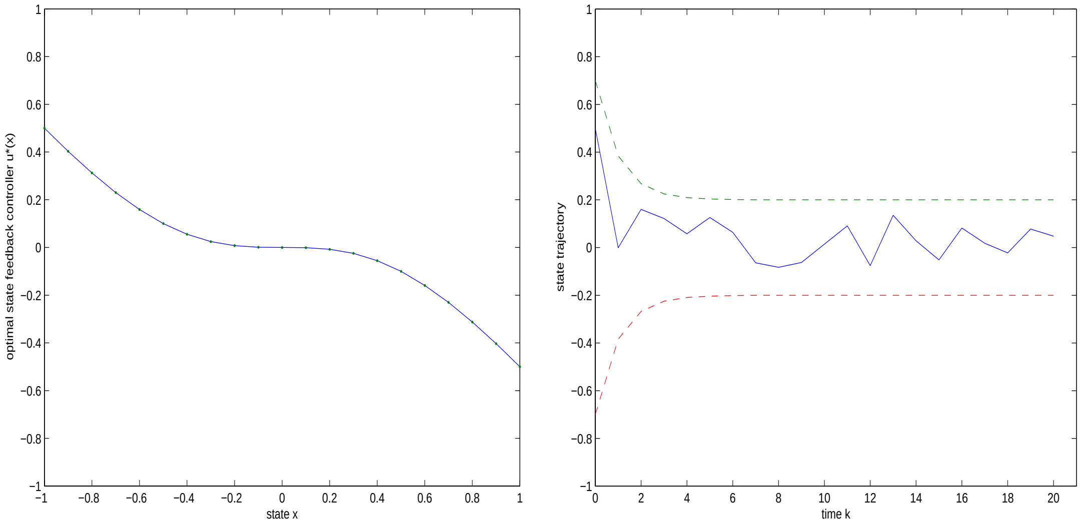


(c) Value function $V_a$: $\eta = 0.5$ (d) optimal controller $u^*$: $\eta = 0.5$.



(e) Value function $V_a$: $\eta = 1$ (f) optimal controller $u^*$: $\eta = 1$.

Figure 2: value function and optimal controller



(a) state feedback controller $u^*(x)$ (b) a trajectory of the closed-loop system

Figure 3: state feedback controller and a trajectory of closed-loop system

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# A State Feedback Uniform LIB Synthesis Results

We provide an outline of the state feedback synthesis results about uniform LIB dissipation. The results listed below are special cases of the general results in [5].

Consider system (4), (7) and the SFULIB problem defined in Section 3. Define the *value function*

$$V_a(x) = \inf_{K\in\mathcal{K}_{state}} \sup_{k\geq 0} \sup_{w_{0,k-1}\in\mathcal{W}_{0,k-1}} \{g(x_k) : u = K(x), x_0 = x\},\ \forall x \in \mathbf{R}^n \tag{42}$$

and denote

$$\text{dom}V_a = \{x \in \mathbf{R}^n : V_a(x) < +\infty\}. \tag{43}$$

Let $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and $S \subseteq \text{dom}V \subseteq \mathbf{R}^n$. The dynamic programming equation (DPE) is

$$V(x) = \max\{g(x), \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} V(f(x,u,w))\},\ \forall x \in S, \tag{44}$$

The analogous dynamic programming inequality (DPI) is

$$V(x) \geq \max\{g(x), \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} V(f(x,u,w))\},\ \forall x \in S. \tag{45}$$

**Theorem A.1** *[5] (Necessity) If there exists a state feedback controller $K_0 \in \mathcal{K}_{state}$ such that the closed-loop system (with $K = K_0$) is uniform LIB dissipative with respect to $B_0$ and $\lambda$, then the function $V_a : \mathbf{R}^n \to \bar{\mathbf{R}}$ defined by (42) satisfies (i)$B_0 \subseteq \text{dom}V_a$, $\sup_{x\in B_0} V_a(x) \leq \lambda$; (ii) $V_a$ solves the DPE (44), with $S = \text{dom}V_a$.*

**Definition A.2** *[5] Given a function $V : \mathbf{R}^n \to \bar{\mathbf{R}}$ and a nonempty set $S \subseteq \text{dom}V \subseteq \mathbf{R}^n$, the pair $(V,S)$ is called a* good solution *of the DPI (45) if it satisfies (i) $(V,S)$ is a solution of the DPI (45) and there exists $\mathbf{u}^* : S \to \mathbf{U}$ such that*

$$\begin{aligned}&\max\{g(x), \sup_{w\in\mathbf{W}} V(f(x,\mathbf{u}^*(x),w))\}\\ =\ &\max\{g(x), \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} V(f(x,u,w))\}, \forall x \in S.\end{aligned} \tag{46}$$

*(ii) $S$ is an invariant set under the closed-loop dynamics when the controller is $\mathbf{u}^*(x)$, i.e. if $x_0 \in S$, then $\forall k \geq 0, \forall w_{0,k-1} \in \mathcal{W}_{0,k-1},\ x_k \in S$, where $x_{i+1} = f(x_i, \mathbf{u}^*(x_i), w_i)$.*

A controller $K^* \in \mathcal{K}_{state}$ can be defined by

$$u_k = K^*(x_{0,k}) = \mathbf{u}^*(x_k),\quad \forall x_k \in \mathbf{R}^n. \tag{47}$$

(If $S \neq \mathbf{R}^n$ we specify $\mathbf{u}^*(x)$ arbitrarily for $x \notin S$). Notice that $K^*$ is a static state feedback controller.

**Theorem A.3** *[5] (Sufficiency) If $(V,S)$ is a good solution of the DPI (45), then the closed-loop system consisting (4), (7) with (47)) satisfies*

$$z_k \leq V(x_0),\ \forall x_0 \in S, \forall w_{0,k-1} \in \mathcal{W}_{0,k-1}, \forall k \geq 0.$$

*Moreover, if $B_0 \subseteq S$ and $\sup_{x\in B_0} V(x) < +\infty$, then the closed-loop system is uniform LIB dissipative with respect to $B_0$ and $\lambda = \sup_{x\in B_0} V(x)$.*