

H^∞ Control for Discrete-time Nonlinear Switching Systems*

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Abstract

This paper formulates and solves the robust H^∞ control problem for discrete time nonlinear switching systems. The H^∞ control problem is interpreted as the l_2 finite gain control problem and is studied using a dissipative systems theory for switched systems. Both state and measurement feedback control problems are formulated as dynamic games and solved using dynamic programming. The partially observed dynamic game corresponding to the measurement feedback control problem is solved by transforming into a completely observed, full state infinite dimensional game problem using information states. Our results are illustrated with an example.

Keywords: Switching systems, H^∞ control, dissipative systems, information state, dynamic games, dynamic programming.

1 Introduction

Switching plays a central role in many practical complex dynamical systems such as chemical processes, automotive processes, electrical systems. The study of switching systems recently has attracted much attention among the control scientists and engineers [1], [3], [15], [16] and a theory of switched systems is being developed. A switched system consists of multiple subsystems and a mechanism for switching between them. Since the subsystems may be subject to uncertainty and external disturbances, the problem of robust control system design for switched systems is of importance.

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One approach to robust control system design is via an l_2 gain criteria. In linear systems context, this problem is known as the H^∞ robust control problem and has been the vast amount of research in control theory for past two decades [4], [23], [6] and the numerous references therein. This problem was generalized to nonlinear systems in [9], [7], [13], [14], [10], [18]. The nonlinear H^∞ robust control can be formulated and solved in the framework of dissipative systems theory, game theory and information states [2], [7], [10], [20].

To date, little research [3], [21] has been reported to study the H^∞ control problem for hybrid systems in general, nor for the important subclass of switching systems. One of the difficulties in such a generalization is in the formulation of a meaningful l^2 gain for a system involving multiple subsystems. Developing a dissipative systems theory for switching systems [22] inevitably involves the multiple storage functions which is much more difficult to deal with than the case of a single storage function.

In this paper, we formulate and solve a robust optimal H^∞ control problem for discrete time nonlinear switching systems. Both state feedback and measurement feedback problems are considered. Following the general nonlinear H^∞ control framework, we interpret the switching H^∞ control problem to be the problem of achieving a desired l_2 finite gain from the disturbances of the subsystems to a performance output. The performance output captures both subsystem performance and the switching costs. By developing a version of dissipative systems theory for switching systems, the finite gain analysis and state feedback control problems are reduced to solving the dissipative equations (inequalities) satisfied by the storage functions. In the case of measurement feedback, we have to construct an information state observer to represent information relevant to the control problem and from which the feedback control can be determined. The solution is expressed in terms of necessary and sufficient conditions. In addition, a certainty equivalence principle is also given.

This paper is organized as follows. Section 2 describes the model of autonomous switched systems considered in this paper and defines and analyzes the dissipation property. Section 3 formulates and the state feedback H^∞ control problem for nonlinear switching control systems and provides the solution in terms of dynamic programming equations (inequalities). The measurement feedback H^∞ control problem is studied in Section 4. In Section 5, an example is given to illustrate the controller design approaches in this paper. Section 6 concludes the paper.

Below are some notations used in this paper:

$$\begin{aligned} w_{0,k-1} &= \{w_0, \dots, w_{k-1}\}, \forall k \geq 1, \\ \mathcal{W}_{0,k-1} &= \{w_{0,k-1} : w_i \in W, 0 \leq i \leq k-1\}, \forall k \geq 1, \\ \mathcal{W}_{0,\infty} &= \{w_{0,\infty} : w_i \in W\}, \\ \bar{\mathbf{R}} &= \mathbf{R} \cup \{+\infty\}, \\ \tilde{\mathbf{R}} &= \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}. \end{aligned}$$

2 Finite Gain Analysis

In this section we present some analysis tools adapting the framework of dissipative systems theory to switched systems. In addition to their utility for the analysis of switched systems, the tools will be used in following sections for controller synthesis.

2.1 System Dynamics

Consider an autonomous switched system G consisting of N subsystems

$$G_i : \begin{cases} x_{k+1} = f_i(x_k, w_k), \\ z_k = g_i(x_k). \end{cases} \quad (1)$$

Here let the subsystems are indexed by a finite set $\mathcal{I} = \{1, 2, \dots, N\}$. We take $x \in \mathbf{R}^n$, $w \in \mathbf{R}^s$ the state and disturbance input for every subsystem. $z \in \mathbf{R}^r$ is the performance measure which is defined appropriately for the particular problem at hand.

Assumption 2.1 *There is at least one equilibrium $(x_e, i_e) \in \mathbf{R}^n \times \mathcal{I}$, that is $x_e = f_{i_e}(x_e, 0)$, $0 = g_{i_e}(x_e)$.*

The switches in an autonomous switched system are autonomous switches produced by a particular switching law (that is fixed or has been designed). There are many different ways to express the switching law. In this paper, we consider the case when the switching law is designated by a pair (ϕ, A)

$$\phi(x, i) = \begin{cases} j(x, i) \neq i, & \text{if } (x, i) \in A, \\ i, & \text{if } (x, i) \notin A, \end{cases} \quad (2)$$

where $A \subset \mathbf{R}^n \times \mathcal{I}$ is the switching set designating when a switch occurs. $\phi : \mathbf{R}^n \times \mathcal{I} \rightarrow \mathcal{I}$ is the switching function telling how a switch actually occurs. It is natural to assume $(x_e, i_e) \notin A$.

Without loss of generality, we assume there is no more than one switch at each time step. In the case when the switching law produce more than one switch for a particular state (x, i) , e.g. switch to (x, j_1) and then switch to (x, j_2) , it is equivalent to switch from (x, i) directly to (x, j_2) and we regard $\phi(x, i) = j_2$.

We assume that the switch does not take any time and the continuous state x does not change when switching (a typical assumption in switching systems). That is, if there is a switch from subsystem i to j immediately after some time $k \geq 0$, then this does not cost any time, and the subsequent evolution of the state takes one step time following the dynamics of subsystem j , $x_{k+1} = f_j(x_k, w_k)$. In another words, at each time step, there are two actions – first switch to a new subsystem (or keep the same subsystem) and this does not consume time, then do a continuous evolution (for one time step).

Now with the switching law ϕ given, the switching system dynamics is expressed by

$$\begin{cases} i_{k+1} = \phi(x_k, i_k), \\ x_{k+1} = f_{i_{k+1}}(x_k, w_k), \\ z_k = g_{i_{k+1}}(x_k). \end{cases} \quad (3)$$

For any initial state (x_0, i_0) , any disturbance sequence $w_{0,k-1}$, a state trajectory $x_{0,k}$ and a performance sequence $z_{0,k-1}$ are determined.

2.2 Dissipation and Storage Functions

Assuming there is a switching cost associated with each switch.

$$\rho(x, i, j) \geq 0 \quad (4)$$

is used to denote the switching cost when switching from subsystem i to subsystem j and the continuous state is at x . Without loss of generality, we denote $\rho(x, i, i) = 0, \forall x \in \mathbf{R}^n, \forall i \in \mathcal{I}$ by which we mean that there is no cost for non-switching.

The following definition can be regarded as a generalization of l_2 finite gain notion for single nonlinear system.

Definition 2.2 *The switching system given in (3) has generalized l_2 finite gain at most $\gamma > 0$ if there exist non-negative functions $\beta^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ satisfying $\beta^{i_e}(x_e) = 0$ such that*

$$\sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) \leq \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 + \beta^i(x) \quad (5)$$

for any $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i)$ and any disturbance sequence $w_{0,k-1}$.

In an open loop switching system, switching is a type of control action. With a positive switching cost, we discourage switching when trying to maintain a small gain with respect to disturbances. For a closed loop switched system, switching is autonomous and is the discrete part of the hybrid dynamics. The left hand of the gain inequality (5) is the hybrid cost penalize both continuous evolutions and discrete transitions caused by the external disturbances. This generalized l^2 gain formulation is similar to that defined by Ball, Chudoung and Day [3].

The functions β^i in Definition 2.2 are called bias functions which take care of the effect due to the initial state. It must be non-negative because (5) needs to hold for zero sequence $w_{0,k-1}$. The above property is some sort of input-output gain property, it is well known that such properties can be studied through some internal energy like quantities which are called storage functions.

Definition 2.3 *A family of non-negative functions $V^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ are called storage functions of the switching system given in (3) if $V^{i_e}(x_e) = 0$ and*

$$V^i(x) \geq \sum_{l=0}^{k-1} (|z_l|^2 - \gamma^2 |w_l|^2) + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) + V^{i_k}(x_k) \quad (6)$$

for any $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i) \in \mathbf{R}^n \times \mathcal{I}$, and any disturbance sequence $w_{0,k-1}$.

Define the available storage functions $V_a^i : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}, i \in \mathcal{I}$ by

$$V_a^i(x) = \sup_{k \geq 0} \sup_{w_{0,k-1}} \left\{ \sum_{l=0}^{k-1} (|z_l|^2 - \gamma^2 |w_l|^2) + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) : x_0 = x, i_0 = i \right\}. \quad (7)$$

By using standard dynamic programming methods, we can obtain the dynamic programming principle for V_a^i . For any $0 \leq k$

$$V_a^{i_0}(x_0) = \sup_{w_0, k-1} \left\{ \sum_{l=0}^{k-1} (|z_l|^2 - \gamma^2 |w_l|^2) + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) + V_a^{i_k}(x_k) \right\}. \quad (8)$$

Equation (8) shows that $V_a^i, i \in \mathcal{I}$ are storage functions when they are finite.

Now we express the l_2 finite gain property given in Definition 2.2 in terms of storage functions.

Theorem 2.4 *The switching system given in (3) has l_2 finite gain at most γ according to Definition 2.2 if and only if there exist storage functions $V^i, i \in \mathcal{I}$ as defined in Definition 2.3.*

PROOF. If the switching system has generalized l_2 finite gain at most γ with bias functions $\beta^i, i \in \mathcal{I}$, then Definition 2.2 implies the available storage functions $V_a^i \leq \beta^i$ are finite, hence they are storage functions. Conversely, if there are storage functions, then they should be larger than the available storage functions V_a^i which can be taken as bias functions on Definition 2.2. □

2.3 Dynamic Programming Equations and Inequalities (DPE and DPI)

The storage functions are defined by the inequality (6), but actually it is enough to define storage functions by considering one-step form of (6) which are the Dynamic programming inequalities (DPI):

$$V^i(x) \geq \sup_{w \in \mathbf{R}^s} \{ |g_{\phi(x,i)}(x)|^2 - \gamma^2 |w|^2 + \rho(x, i, \phi(x, i)) + V^{\phi(x,i)}(f_{\phi(x,i)}(x, w)) \}. \quad (9)$$

Or equivalently (write the switching and non-switching separately)

$$\begin{cases} V^i(x) \geq \sup_{w \in \mathbf{R}^s} \{ |g_{j(x,i)}(x)|^2 - \gamma^2 |w|^2 + \rho(x, i, j(x, i)) + V^{j(x,i)}(f_{j(x,i)}(x, w)) \}, & (x, i) \in A, \\ V^i(x) \geq \sup_{w \in \mathbf{R}^s} \{ |g_i(x)|^2 - \gamma^2 |w|^2 + V^i(f_i(x, w)) \}, & (x, i) \notin A. \end{cases} \quad (10)$$

Theorem 2.5 $V^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ are storage functions according to Definition 2.3 if and only if $V^{i_e}(x_e) = 0, i \in \mathcal{I}$ and they solve inequalities (9).

The Dynamic Programming Equations (DPE) are the one step form of the dynamic programming principle of the available storage functions (8)

$$V^i(x) \sup_{w \in \mathbf{R}^s} \{ |g_{\phi(x,i)}(x)|^2 - \gamma^2 |w|^2 + \rho(x, i, \phi(x, i)) + V^{\phi(x,i)}(f_{\phi(x,i)}(x, w)) \}. \quad (11)$$

The following two theorems are direct from Definition 2.2, 2.3 and Theorem 2.4.

Theorem 2.6 (Necessity) *If the switching system given in (3) has l_2 finite gain at most γ according to Definition 2.2, then the available storage functions defined by (7) are finite everywhere, $V_a^{i_e}(x_e) = 0$, and satisfy the DPE (11).*

Theorem 2.7 (Sufficiency) *If there exist functions $V^i, i \in \mathcal{I}$ such that $V^{i_e}(x_e) = 0$ and DPI (9) hold, then the switching system given in (3) has l_2 finite gain at most γ according to Definition 2.2.*

3 State Feedback Control

This section studies the problem of finding state feedback controllers for a switching system such that the closed-loop system has generalized finite l_2 gain.

3.1 State Feedback Control Problem

Suppose a switching system G consists of N subsystems

$$G_i : \begin{cases} x_{k+1} = f_i(x_k, u_k, w_k), \\ z_k = g_i(x_k, u_k). \end{cases} \quad (12)$$

Here let the subsystems are indexed by a finite set $\mathcal{I} = \{1, 2, \dots, N\}$. We take $x \in \mathbf{R}^n$, $u \in U \subset \mathbf{R}^m$, $w \in \mathbf{R}^s$ the state, control input, and disturbance input for every subsystem. $z \in \mathbf{R}^r$ is the performance measure. We assume there is an equilibrium point (x_e, i_e) such that $f_{i_e}(x_e, 0, 0) = 0$ and $g_{i_e}(x_e, 0) = 0$.

For the control problem, at each time step, besides the control value u_k , we also need to decide which subsystem to switch to (or keep the current subsystem). So the control takes values in the set $\mathcal{I} \times U$. We first define the admissible state feedback controller.

Definition 3.1 *An admissible state feedback controller K is a causal map that maps a state sequence into a control pair. To state it clearly,*

$$K(x_{0,k}, i_{0,k}) = (i_{k+1}, u_k) \in \mathcal{I} \times U, \quad k = 0, 1, \dots \quad (13)$$

Denote by \mathcal{K}_{sf} the class of such admissible state feedback controllers.

We require that the state feedback controller to be null initialized, that means for any sequence consisting of equilibrium points $(x_{0,k}, i_{0,k}), (x_l, i_l) = (x_e, i_e), \forall 0 \leq l \leq k, k \geq 0$, it holds $K(x_{0,k}, i_{0,k}) = (i_e, 0)$. Under a null initialized state feedback controller, the equilibrium point of the open loop system (x_e, i_e) is also the equilibrium point of the closed loop system.

Suppose an admissible state feedback controller K is designed. Then for any given initial state $(x_0, i_0) \in \mathbf{R}^n \times \mathcal{I}$, any disturbances sequence $w_{0,\infty}$, a trajectory of the closed loop system, $(x_{0,\infty}, i_{0,\infty})$, can be obtained as follows:

$$\begin{aligned} (i_1, u_0) &= K(x_0, i_0), & x_1 &= f_{i_1}(x_0, u_0, w_0); \\ (i_2, u_1) &= K(x_0, i_0, x_1, i_1), & x_2 &= f_{i_2}(x_1, u_1, w_1); \\ &\dots & &\dots \end{aligned} \quad (14)$$

Problem: For a fixed gain $\gamma > 0$, find a state feedback controller (if exists) $K \in \mathcal{K}_{sf}$ such that the closed-loop system has generalized l_2 finite gain at most γ .

3.2 Solution to The State Feedback Synthesis Problem

The state feedback control problem is solved through a dynamic game between the disturbance and controller. In this dynamic game, the disturbance is regarded as the maximizer and the controller as the minimizer. The value functions $\mathbf{V}_a^i : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}, i \in \mathcal{I}$ are defined as

$$\mathbf{V}_a^i(x) = \inf_{K \in \mathcal{K}_{sf}} \sup_{k \geq 0} \sup_{w_0, k-1} \left\{ \sum_{l=0}^{k-1} (|z_l|^2 - \gamma^2 |w_l|^2) + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) \right\}. \quad (15)$$

The solution of this dynamic game problem is to find (if exists) the optimal strategy $K^* \in \mathcal{K}_{sf}$ for the minimizer achieving minimum at the right hand side of (15). As we shall see K^* solves the l_2 finite gain state feedback control problem.

The dynamic game is solved by the approach of the dynamic programming. By dynamic programming argument, we establish the Dynamic Programming Principle for the value functions \mathbf{V}^i :

$$\mathbf{V}^{i_0}(x_0) = \inf_{K \in \mathcal{K}_{sf}} \sup_{w_0, k-1} \left\{ \sum_{l=0}^{k-1} (|z_l|^2 - \gamma^2 |w_l|^2) + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) + \mathbf{V}^{i_k}(x_k) \right\}. \quad (16)$$

The one-step relation in (16) are the equations that we use to solve the game (hence l_2 finite gain state feedback control) problem

$$\mathbf{V}^i(x) = \inf_{(j,u) \in \mathcal{I} \times U} \sup_w \{ |g_j(x, u)|^2 - \gamma^2 |w|^2 + \rho(x, i, j) + \mathbf{V}^j(f_j(x, u, w)) \}. \quad (17)$$

If we want to state the switching and non-switching clearly, then

$$\mathbf{V}^i(x) = \min \left\{ \begin{array}{l} \inf_{(j,u) \in \mathcal{I} \times U, j \neq i} \sup_w \{ |g_j(x, u)|^2 - \gamma^2 |w|^2 + \rho(x, i, j) + \mathbf{V}^j(f_j(x, u, w)) \}, \\ \inf_{u \in U} \sup_w \{ |g_i(x, u)|^2 - \gamma^2 |w|^2 + \rho(x, i, i) + \mathbf{V}^i(f_i(x, u, w)) \}. \end{array} \right\} \quad (18)$$

If the l_2 finite gain state feedback control problem is solved by some $K \in \mathcal{K}_{sf}$, then obviously the value functions $\mathbf{V}^i, i \in \mathcal{I}$ defined in (15) are finite everywhere, and solve the equations (17), so we obtain the necessary condition of l_2 finite gain state feedback synthesis problem.

Theorem 3.2 (Necessity) *If some $K_0 \in \mathcal{K}_{sf}$ solves the l_2 finite gain state feedback control problem, then the value functions $\mathbf{V}_a^i, i \in \mathcal{I}$ defined by (15) are finite everywhere, satisfy $\mathbf{V}_a^{i_e}(x_e) = 0$ and the DPE (17).*

To obtain an l_2 finite gain state feedback controller, all we need is a set of solutions $\mathbf{V}^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ to the inequalities

$$\mathbf{V}^i(x) \geq \inf_{(j,u) \in \mathcal{I} \times U} \sup_w \{ |g_j(x, u)|^2 - \gamma^2 |w|^2 + \rho(x, i, j) + \mathbf{V}^j(f_j(x, u, w)) \}. \quad (19)$$

With such a solution, we can obtain a static state feedback controller K^* by

$$K^*(x, i) = (j^*(x, i), u^*(x, i)), \quad (20)$$

where (j^*, u^*) is the pair that achieves the infimum in (19). The associated switching set of the closed loop switched system can be obtained either by

$$A^* = \{(x, i) \in \mathbf{R}^n \times \mathcal{I} | j^*(x, i) \neq i\} \quad (21)$$

or by

$$A^* = \left\{ (x, i) \in \mathbf{R}^n \times \mathcal{I} \mid \inf_{u \in U} \sup_w \{ |g_i(x, u)|^2 - \gamma^2 |w|^2 + \mathbf{V}^i(f_i(x, u, w)) \} \right. \\ \left. > \inf_{(j, u) \in \mathcal{I} \times U, j \neq i} \sup_w \{ |g_j(x, u)|^2 - \gamma^2 |w|^2 + \rho(x, i, j) + \mathbf{V}^j(f_j(x, u, w)) \} \right\}. \quad (22)$$

Theorem 3.3 (Verification) *Let $\mathbf{V}^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ solve the inequalities (19), then the state feedback controller K^* determined by (20) is an l_2 finite gain state feedback controller.*

The verification theory can be proved by standard dynamic programming argument [11],[12].

Remark 3.4 When a state feedback controller is obtained from a solution of the equations (17), then the controller is called the optimal H^∞ state feedback controller.

4 Measurement Feedback H^∞ Control

This section studies the problem of finding measurement feedback controllers for a switching system such that the closed-loop system has generalized finite l_2 gain.

4.1 Measurement Feedback Control Problem

Consider a switching system G consists of N subsystems

$$G : \begin{cases} x_{k+1} = f_i(x_k, u_k) + w_k, & k \geq 0, \\ y_k = h_i(x_k) + v_k, & k \geq 1, \\ z_k = g_i(x_k, u_k), & k \geq 0. \end{cases} \quad (23)$$

Here the state x is not available in general, instead a measurement output $y \in \mathbf{R}^p$ is observed, $v \in \mathbf{R}^p$ is the disturbance in the observation process. We assume there is an equilibrium point (x_e, i_e) such that $f_{i_e}(x_e, 0) = 0$, $h_{i_e}(x_e) = 0$ and $g_{i_e}(x_e, 0) = 0$.

At any time step, one of the N subsystems is active and the rest subsystems are kept silent. At some particular switching time, the active subsystem is switched to another subsystem. We assume that the index of the active subsystem is always available to the designer though the state x is not directly known.

Similar to the state feedback control problem, at each time step, besides the control value u_k , we also need to decide which subsystem to switch to (or keep the current subsystem). So the control takes values in the set $\mathcal{I} \times U$. We first define the admissible measurement feedback controller.

Definition 4.1 An admissible measurement feedback controller K is a causal map that maps a measurement sequence into a control pair. To state it clearly,

$$K(i_0, y_{1,k}) = (i_{k+1}, u_k) \in \mathcal{I} \times U, \quad k = 0, \dots \quad (24)$$

Denote by \mathcal{K}_{mf} the class of such admissible measurement feedback controllers.

Note the first control (i_1, u_0) is produced by K but it does not depend on the observation y_k for any $k \geq 1$. It depends on i_0 . Similar with the state feedback controller, we require the measurement feedback controller to be null initialized.

Suppose an admissible measurement feedback controller K is designed. Then for any given initial state $(x_0, i_0) \in \mathbf{R}^n \times \mathcal{I}$, any observation disturbances sequence $v_{1,\infty}$ and any dynamics disturbances sequence, $w_{0,\infty}$, a trajectory of the closed loop system, $(x_{0,\infty}, i_{0,\infty})$, can be determined as follows:

$$\begin{aligned} (i_1, u_0) &= K(i_0); \\ x_1 &= f_{i_1}(x_0, u_0) + w_0, \\ y_1 &= h_{i_1}(x_1) + v_1, \quad (i_2, u_1) = K(i_0, y_1); \\ x_2 &= f_{i_2}(x_1, u_1) + w_1, \\ y_2 &= h_{i_2}(x_2) + v_2, \quad (i_3, u_2) = K(i_0, y_1, y_2); \\ &\vdots \end{aligned} \quad (25)$$

Problem: Find a measurement feedback controller $K \in \mathcal{K}_{mf}$ such that the closed-loop system has l_2 finite gain at most γ . That is, there exist functions $\beta^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ satisfying $\beta^{i_e}(x_e) = 0$ such that

$$\sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) \leq \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 + \gamma^2 \sum_{l=1}^k |v_l|^2 + \beta^i(x) \quad (26)$$

for any $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i)$ and any disturbance sequence $w_{0,k-1}$ and $v_{1,k}$. Here $\rho(x, i, j)$ is the switching cost defined by (4).

Let's denote the set of all bias functions with initial subsystem $i \in \mathcal{I}$ and controller $K \in \mathcal{K}_{mf}$ to be \mathcal{B}_K^i . The minimum bias function is denoted by β_K^i defined as

$$\beta_K^i(x) = \sup_{k \geq 0} \sup_{w_{0,k-1}} \sup_{v_{1,k}} \left\{ \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2 : x_0 = x, i_0 = i \right\}.$$

4.2 An Information State Formulation

Denote $\tilde{\mathcal{X}} = \{p : \mathbf{R}^n \rightarrow \tilde{\mathbf{R}}\}$. For $p, q \in \tilde{\mathcal{X}}$, denote $\langle p \rangle = \sup_{x \in \mathbf{R}^n} p(x)$ and $(p, q) = \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}$.

For any fixed admissible controller $K \in \mathcal{K}_{mf}$, define the cost for the closed loop system (G, K) with fixed $p \in \tilde{\mathcal{X}}$ and $i_0 \in \mathcal{I}$ by

$$\mathbf{J}_p^{i_0}(K) = \sup_{k \geq 0} \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k-1}} \sup_{v_{1,k}} \left\{ p(x_0) + \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2 \right\}. \quad (27)$$

The cost \mathbf{J} encodes the finite gain condition for the closed loop system (G, K) as indicated in the following proposition.

Proposition 4.2 *The closed-loop system (G, K) has finite l_2 gain less than γ if and only if there exist functions $p^i \in \tilde{\mathcal{X}}$ with $-\infty < p^i(x) < +\infty, \forall x \in \mathbf{R}^n, \forall i \in \mathcal{I}, p^{i_e}(x_e) = 0$ such that*

$$\mathbf{J}_{p^i}^i(K) \leq 0, \quad \forall i \in \mathcal{I}.$$

PROOF. Assume the closed-loop system (G, K) has finite l_2 gain less than γ . That is, there are functions $\beta^i : \mathbf{R}^n \rightarrow \mathbf{R}, i \in \mathcal{I}$ satisfying $\beta^{i_e}(x_e) = 0$ such that

$$\sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) \leq \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 + \gamma^2 \sum_{l=1}^k |v_l|^2 + \beta^i(x)$$

for any $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i)$ and any disturbance sequence $w_{0,k-1}$ and $v_{1,k}$. Take $p^i(x) = -\beta^i(x), \forall x \in \mathbf{R}^n, \forall i \in \mathcal{I}$, we get

$$p^i(x) + \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2 \leq 0$$

for any $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i)$ and any disturbance sequence $w_{0,k-1}$ and $v_{1,k}$. That is

$$\mathbf{J}_{p^i}^i(K) \leq 0.$$

Conversely, assume there exist $p^i \in \tilde{\mathcal{X}}$ with $-\infty < p^i(x) < +\infty, \forall x \in \mathbf{R}^n, \forall i \in \mathcal{I}$ satisfying $p^{i_e}(x_e) = 0$ such that $\mathbf{J}_{p^i}^i(K) \leq 0$. Then for any $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i)$ and any disturbance sequence $w_{0,k-1}$ and $v_{1,k}$, it holds

$$\sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) \leq \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 + \gamma^2 \sum_{l=1}^k |v_l|^2 - p^i(x).$$

Take $\beta^i(x) = -p^i(x), (x, i) \in \mathbf{R}^n \times \mathcal{I}$, we have shown that (G, K) has l_2 finite gain at most γ . □

Now the H^∞ control problem requires finding a controller $K \in \mathcal{K}_{mf}$ such that $\mathbf{J}_{p^i}^i(K) \leq 0, \forall i \in \mathcal{I}$ for some $p^i \in \tilde{\mathcal{X}}$ with $-\infty < p^i(x) < +\infty, \forall x \in \mathbf{R}^n, i \in \mathcal{I}$ satisfying $p^{i_e}(x_e) = 0$. The essential difficulty in doing so is that the cost $\mathbf{J}_{p^i}^i(K)$ is expressed in terms of the state x (via z, ρ), while K can only use the information available in the observed process y from which we don't know the exact value of the state x . This situation is well known in control problems for partially observable stochastic systems. The problem is solved by constructing appropriate "information state" which is a dynamic quantity determined by the observations and contains the information of the state relevant to the control problem. Similar idea was transported to deterministic control systems by James and Baras [12], [10] in solving the general partially observed dynamic game problems. By constructing an appropriate information state, the cost is equivalently expressed in terms of the observation via this information state. Then the original measurement feedback control problem

is transformed into a state feedback control problem with the information state as the new controlled state. We employ a similar idea here to treat our H^∞ control problem for switched systems. The main thing here is to construct an appropriate information state and determines its dynamics.

Definition 4.3 Given the initial discrete state $i_0 \in \mathcal{I}$, the observation sequence $y_{1,k}$, the switched control signal $(i_1, u_0), (i_2, u_1), \dots, (i_k, u_{k-1})$ and $p_0 \in \tilde{\mathcal{X}}$, define the information state $p_k \in \tilde{\mathcal{X}}$ to be

$$p_k(x) = \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k-1}} \sup_{v_{1,k}} \{p_0(x_0) + \sum_{l=1}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2 \\ : x_k = x, x_{l+1} = f_{i_{l+1}}(x_l, u_l) + w_l, 0 \leq l \leq k-1; h_{i_l}(x_l) + v_l = y_l, 1 \leq l \leq k\} \quad (28)$$

where the constraints means the state trajectories $x(\cdot)$ of the switched system G under the control sequence and disturbance sequences $w(\cdot), v(\cdot)$ satisfies the terminal state constraints $x_k = x$ and produces observation signal $y(\cdot)$ on $[1, k]$.

The information state represents the worst case l_2 gain cost of those state trajectories consistent with the observation. To obtain the dynamics for the information state p_k , we define $F_j(p, i, u, y) \in \tilde{\mathcal{X}}$ by

$$F_j(p, i, u, y)(x) = \sup_{\xi \in \mathbf{R}^n} \{p(\xi) + B_j(\xi, x, i, u, y)\} \quad (29)$$

where

$$B_j(\xi, x, i, u, y) = |g_j(\xi, u)|^2 + \rho(\xi, i, j) - \gamma^2 |x - f_j(\xi, u)|^2 - \gamma^2 |y - h_j(x)|^2.$$

Lemma 4.4 Given initial subsystem $i_0 \in \mathcal{I}$, the observation sequence y_1, y_2, \dots , the switched control signal $(i_1, u_0), (i_2, u_1), \dots$ and $p_0 \in \tilde{\mathcal{X}}$, the information state defined in Definition 4.3 is the solution of the following recursion

$$\begin{cases} p_k = F_{i_k}(p_{k-1}, i_{k-1}, u_{k-1}, y_k), k \geq 1, \\ p_0(x) \in \tilde{\mathcal{X}}. \end{cases} \quad (30)$$

PROOF. For $k \geq 1$

$$\begin{aligned} & F_{i_k}(p_{k-1}, i_{k-1}, u_{k-1}, y_k)(x) \\ &= \sup_{\xi \in \mathbf{R}^n} \{p_{k-1}(\xi) + B_{i_k}(\xi, x, i_{k-1}, u_{k-1}, y_k)\} \\ &= \sup_{\xi \in \mathbf{R}^n} \{p_{k-1}(\xi) + |g_{i_k}(\xi, u_{k-1})|^2 + \rho(\xi, i_{k-1}, i_k) - \gamma^2 |x - f_{i_k}(\xi, u_{k-1})|^2 - \gamma^2 |y_k - h_{i_k}(x)|^2\} \\ &= \sup_{\xi \in \mathbf{R}^n} \{ \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k-2}} \sup_{v_{1,k-1}} \{p(x_0) + \sum_{l=0}^{k-2} |z_l|^2 + \sum_{l=0}^{k-2} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-2} |w_l|^2 - \gamma^2 \sum_{l=1}^{k-1} |v_l|^2 \\ &\quad : x_{k-1} = \xi, x_{l+1} = f_{i_{l+1}}(x_l, u_l) + w_l, 0 \leq l \leq k-2; h_{i_l}(x_l) + v_l = y_l, 1 \leq l \leq k-1\} \\ &\quad + |g_{i_k}(\xi, u_{k-1})|^2 + \rho(\xi, i_{k-1}, i_k) - \gamma^2 |w_{k-1}|^2 - \gamma^2 |v_k|^2 \} \\ &= \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k-1}} \sup_{v_{1,k}} \{p(x_0) + \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2 \\ &\quad : x_k = x, x_{l+1} = f_{i_{l+1}}(x_l, u_l) + w_l, 0 \leq l \leq k-1; h_{i_l}(x_l) + v_l = y_l, 1 \leq l \leq k\} \\ &= p_k(x). \end{aligned}$$

□

Now given $i_0 = i \in \mathcal{I}$ and let the input switching signal $(i_1, u_0), (i_2, u_1), \dots$, is related with the measurement signal y_1, y_2, \dots by an output feedback controller $K \in \mathcal{K}_{mf}$ by (24), we define a new cost which is a function of the information state p_k (hence the measurement signal y .)

$$\bar{\mathbf{J}}_p^i(K) = \sup_{k \geq 0} \sup_{y_{1,k}} \{ \langle p_k \rangle : p_0 = p, i_0 = i \}. \quad (31)$$

Proposition 4.5

$$\mathbf{J}_p^i(K) = \bar{\mathbf{J}}_p^i(K), \forall p \in \tilde{\mathcal{X}}, \forall i \in \mathcal{I}. \quad (32)$$

PROOF. We first show $\bar{\mathbf{J}}_p^i(K) \geq \mathbf{J}_p^i(K)$.

Fix arbitrary $0 \leq k$, any initial hybrid state $(x_0, i_0) = (x, i)$ and any disturbance sequences $w_{0,k-1}$ and $v_{1,k}$. Observation $y_{1,k}$, control sequence $(i_1, u_0), (i_2, u_1), \dots, (i_k, u_{k-1})$ are produced according to the dynamics of the closed-loop system (G, K) . The information state p_k is defined by (4.3) for given $p_0 = p$, so

$$\begin{aligned} \bar{\mathbf{J}}_p^i(K) &\geq \langle p_k \rangle \\ &\geq p_k(x_k) \\ &\geq p(x_0) + \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2. \end{aligned}$$

Hence

$$\begin{aligned} &\bar{\mathbf{J}}_p^i(K) \\ &\geq \sup_{k \geq 0} \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k-1}} \sup_{v_{1,k}} \left\{ p(x_0) + \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2 \right\} \\ &= \mathbf{J}_p^i(K). \end{aligned}$$

We next show the converse inequality $\bar{\mathbf{J}}_p^i(K) \leq \mathbf{J}_p^i(K)$.

Fix any $i_0 = i \in \mathcal{I}$, any $k \geq 0$, any $y_{1,k}$, suppose the switched control signal $(i_1, u_0), (i_2, u_1), \dots, (i_k, u_{k-1})$ is produced by the admissible output feedback controller K . For any $x \in \mathbf{R}^n$, if there exists a state trajectory $(i_0, x_0), (i_1, x_1), \dots, (i_k, x_k)$ consistent with the terminal state, observation and control sequences, i.e. there exist $x_0, w_{0,k-1}, v_{1,k}$ such that

$$x_k = x, x_{l+1} = f_{i_{l+1}}(x_l, u_l) + w_l, 0 \leq l \leq k-1; h_{i_l}(x_l) + v_l = y_l, 1 \leq l \leq k.$$

then we have

$$\mathbf{J}_p^i(K) \geq p(x_0) + \sum_{l=0}^{k-1} |z_l|^2 + \sum_{l=0}^{k-1} \rho(x_l, i_l, i_{l+1}) - \gamma^2 \sum_{l=0}^{k-1} |w_l|^2 - \gamma^2 \sum_{l=1}^k |v_l|^2.$$

Noticing the definition of the information state (4.3), we have

$$\mathbf{J}_p^i(K) \geq p_k(x)$$

For those $x \in \mathbf{R}^n$ that are inconsistent with the observation and control sequences, $p_k(x) = -\infty$ by definition of the information state (4.3). We also have $\mathbf{J}_p^i(K) \geq p_k(x)$.

Thus in both cases, we obtain

$$\mathbf{J}_p^i(K) \geq \sup_{x \in \mathbf{R}^n} p_k(x) = \langle p_k \rangle.$$

Since k is arbitrary, we have

$$\mathbf{J}_p^i(K) \geq \bar{\mathbf{J}}_p^i(K).$$

□

4.3 Solution of The Measurement Feedback Control Problem

Combining Proposition 4.2 and Proposition 4.5, we know the admissible measurement feedback controller $K \in \mathcal{K}_{mf}$ solves the H^∞ control problem for the switched system G if and only if there exist functions $p^i \in \tilde{\mathcal{X}}, i \in \mathcal{I}$ such that $\bar{\mathbf{J}}_{p^i}^i(K) \leq 0$. $\bar{\mathbf{J}}_{p^i}^i(K)$ is an L^∞ bounded (LIB) (see [8]) type performance criteria for a completely observable but infinite dimensional system (30). In this system, the state variable is the information state p_k and the disturbance is the observation signals (y_1, y_2, \dots) . Hence the original H^∞ output feedback control problem is transformed into an equivalent LIB state feedback control problem. After such a transformation, we are able to derive the necessary and sufficient conditions for the original H^∞ measurement feedback control problem and obtain the controllers.

Define the value functions $\mathbf{W}_a^i : \tilde{\mathcal{X}} \rightarrow \tilde{\mathbf{R}}$ by

$$\mathbf{W}_a^i(p) = \inf_{K \in \mathcal{K}_{mf}} \bar{\mathbf{J}}_p^i(K) \quad (33)$$

Note that this is the (lower) value functions of a dynamic game problem where the minimizer (controller) uses Elliott-Kalton strategies [5]. The dynamic programming equation for this value function is given by

$$\mathbf{W}^i(p) = \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \{ \mathbf{W}^j(F_j(p, i, u, y)) \}. \quad (34)$$

For a function $\mathbf{W} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathbf{R}}$, denote

$$\text{dom} \mathbf{W} = \{p \in \tilde{\mathcal{X}} : \mathbf{W}(p) \text{ finite}\}.$$

Theorem 4.6 (Necessity, see Theorem 4.3.1 in [7], Theorem 4.23 in [10]) Assume that $K_0 \in \mathcal{K}_{mf}$ solves the measurement H^∞ control problem with bias function $\beta^i \in \mathcal{B}_K^i$, then the value functions \mathbf{W}_a^i defined by (33) satisfy:

- (i) For any $i \in \mathcal{I}$, $\text{dom} \mathbf{W}_a^i$ is nonempty, $-\beta_{K_0}^i \in \text{dom} \mathbf{W}_a^i$ and $\mathbf{W}_a^{i_e}(-\beta^{i_e}) = 0, \beta^{i_e} \in \mathcal{B}_{K_0}^{i_e}$.
- (ii) Structural properties: For any $i \in \mathcal{I}$,

(iia) \mathbf{W}_a^i dominates $\langle \cdot \rangle$: $\mathbf{W}_a^i(p) \geq \langle p \rangle$, $\forall p \in \tilde{\mathcal{X}}$

(iib) \mathbf{W}_a^i is monotone: if $p_1 \in \text{dom} \mathbf{W}_a^i$ and $p_1 \geq p_2$ such that $\langle p_2 \rangle > -\infty$, then $p_2 \in \text{dom} \mathbf{W}_a^i$ and $\mathbf{W}_a^i(p_1) \geq \mathbf{W}_a^i(p_2)$.

(iic) \mathbf{W}_a^i is additive homogeneous: $\forall c \in \mathbf{R}$, $\mathbf{W}_a^i(p + c) = \mathbf{W}_a^i(p) + c$.

(iii) For any $i \in \mathcal{I}$, $\mathbf{W}_a^i(p) = -\infty$ if and only if $p \equiv -\infty$.

(iv) The dynamic programming programming equation (34) holds.

PROOF. **proof of item (i)**: From the definition of \mathbf{J} in (27), we know

$$\langle p \rangle \leq \mathbf{J}_p^i(K) = \langle p + \beta_K^i \rangle \leq \langle p + \beta^i \rangle, \quad \forall i \in \mathcal{I}, p \in \tilde{\mathcal{X}}, K \in \mathcal{K}_{mf}, \beta^i \in \mathcal{B}_K^i \quad (35)$$

By assumption

$$-\infty < \langle -\beta_{K_0}^i \rangle \leq \mathbf{J}_{-\beta_{K_0}^i}^i(K_0) \leq 0.$$

From Proposition 4.5 and the definition of \mathbf{W}_a^i in (33), we know

$$\langle p \rangle \leq \mathbf{W}_a^i(p) \leq \mathbf{J}_p^i(K_0), \quad (36)$$

hence

$$-\infty < \langle -\beta_{K_0}^i \rangle \leq \mathbf{W}_a^i(-\beta_{K_0}^i) \leq 0.$$

This means $-\beta_{K_0}^i \in \text{dom}(\mathbf{W}_a^i)$. Now for any $\beta^{ie} \in \mathcal{B}_{K_0}^{ie}$, from (35), we have

$$0 = \langle -\beta^{ie} \rangle \leq \mathbf{W}_a^{ie}(-\beta^{ie}) \leq \langle -\beta^{ie} + \beta^{ie} \rangle = 0,$$

which shows $\mathbf{W}_a^{ie}(-\beta^{ie}) = 0$. Then from (35) $\mathbf{J}_p^i(K)$

proof of item (ii):

(iia): It is shown in (36).

(iib): From (35)

$$\begin{aligned} \mathbf{J}_{p_1}^i(K) &= \langle p_1 + \beta_K^i \rangle \\ &\geq \langle p_2 + \beta_K^i \rangle \\ &= \mathbf{J}_{p_2}^i(K). \end{aligned}$$

Hence

$$+\infty > \mathbf{W}_a^i(p_1) \geq \mathbf{W}_a^i(p_2) \geq \langle p_2 \rangle > -\infty.$$

This shows $p_2 \in \text{dom} \mathbf{W}_a^i$.

(iic): This follows from the fact $\mathbf{J}_{p+c}^i(K) = \mathbf{J}_p^i(K) + c$.

proof of item (iii): If $p \equiv -\infty$, then for any $i \in \mathcal{I}$,

$$\mathbf{J}_p^i(K_0) = \langle p + \beta_{K_0}^i \rangle = -\infty,$$

hence

$$\mathbf{W}_a^i(p) \leq \mathbf{J}_p^i(K_0) = -\infty.$$

Conversely, $\langle p \rangle \leq \mathbf{W}_a^i(p) = -\infty$, this means $p \equiv -\infty$.

proof of item (iv):

Let's denote

$$\overline{\mathbf{W}}^i(p) = \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \max \{ \langle p \rangle, \mathbf{W}_a^j(F_j(p, i, u, y)) \}.$$

Fix any $\epsilon > 0$, let $(\bar{i}_1, \bar{u}_0) \in \mathcal{I} \times U$ be such that

$$\overline{\mathbf{W}}^i(p) \geq \max \left\{ \langle p \rangle, \mathbf{W}_a^{\bar{i}_1}(F_{\bar{i}_1}(p, i, \bar{u}_0, y_1)) \right\} - \epsilon$$

where $y_1 \in \mathbf{R}^p$ is arbitrary. Let's denote $p' = F_{\bar{i}_1}(p, i, \bar{u}_0, y_1)$. According to definition (33), there exists $K_1 \in \mathcal{K}_{mf}$ such that

$$\mathbf{W}_a^{\bar{i}_1}(p') \geq \bar{\mathbf{J}}_{p'}^{\bar{i}_1}(K_1) - \epsilon.$$

So it holds

$$\overline{\mathbf{W}}^i(p) \geq \max \left\{ \langle p \rangle, \bar{\mathbf{J}}_{p'}^{\bar{i}_1}(K_1) \right\} - 2\epsilon. \quad (37)$$

Define $K \in \mathcal{K}_{mf}$ by setting

$$K(i_0, y_{1,k}) = \begin{cases} (\bar{i}_1, \bar{u}_0), & k = 0, \\ K_1(\bar{i}_1), & k = 1, \\ K_1(\bar{i}_1, \tilde{y}_{1,k-1}), & k \geq 2. \end{cases}$$

where $\tilde{y}_l = y_{l+1}$, $1 \leq l \leq k-1$. Fix any $k \geq 1$ and $y_{1,k}$, let p_k be the information state determined by $p_0 = p$, $i_0 = i$, $K \in \mathcal{K}_{mf}$, we have

$$p_k = \tilde{p}_{k-1}.$$

where \tilde{p}_k is determined by $K_1 \in \mathcal{K}_{mf}$, $\tilde{y}_{1,k}, \tilde{p}_0 = p', i_0 = \bar{i}_1$. Then

$$\langle p_k \rangle = \langle \tilde{p}_{k-1} \rangle \leq \bar{\mathbf{J}}_{p'}^{\bar{i}_1}(K_1).$$

From (37), we obtain

$$\overline{\mathbf{W}}^i(p) \geq \langle p_k \rangle - 2\epsilon, \quad \forall k \geq 0, y_{1,k}.$$

Thus

$$\overline{\mathbf{W}}^i(p) \geq \bar{\mathbf{J}}_p^i(K) - 2\epsilon \geq \mathbf{W}_a^i(p) - 2\epsilon$$

and

$$\overline{\mathbf{W}}^i(p) \geq \mathbf{W}_a^i(p) \quad \forall p \in \tilde{\mathcal{X}}.$$

But we already know $\mathbf{W}_a^i(p) \geq \langle p \rangle$, we conclude

$$\mathbf{W}_a^i(p) \leq \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \{ \mathbf{W}_a^j(F_j(p, i, u, y)) \}.$$

Now we show the converse inequality.

For any fixed $\epsilon > 0$, let $K_1 \in \mathcal{K}_{mf}$ to be

$$\begin{aligned} \mathbf{W}_a^i(p) &\geq \bar{\mathbf{J}}_p^i(K_1) - \epsilon \\ &= \sup_{k \geq 0} \sup_{y_{1,k}} \{ \langle p_k \rangle : p_0 = p, i_0 = i \} - \epsilon \end{aligned} \quad (38)$$

Let $(\hat{i}_1, \hat{u}_0) = K_1(i)$, together with an arbitrary $\hat{y} \in \mathbf{R}^p$, we obtain

$$\hat{p} = F_{\hat{i}_1}(p, i, \hat{u}_0, \hat{y}).$$

Let $\bar{k} \geq 0$, $\bar{y}_{1, \bar{k}}$ be such that

$$\langle \bar{p}_{\bar{k}} \rangle \geq \mathbf{J}_{\hat{p}}^{\hat{i}_1}(K_1) - \epsilon. \quad (39)$$

Define a particular $y_{1, \bar{k}+1}$ to be

$$y_1 = \hat{y}, y_l = \bar{y}_{l-1}, \quad 1 \leq 2 \leq \bar{k} + 1.$$

Denote $\bar{p}_{\bar{k}}$ to be the information state determined by K_1 , \hat{i}_1 , $\bar{p}_0 = \hat{p}$, $\bar{y}_{1, \bar{k}}$, and $p_{\bar{k}+1}$ be the information state determined by K_1 , i , $p_0 = p$, $y_{1, \bar{k}+1}$. It holds

$$\bar{p}_{\bar{k}} = p_{\bar{k}+1}.$$

From (38), we get

$$\begin{aligned} \mathbf{W}_a^i(p) &\geq \langle p_{\bar{k}+1} \rangle - \epsilon \\ &= \langle \bar{p}_{\bar{k}} \rangle - \epsilon \\ &\geq \mathbf{J}_{\hat{p}}^{\hat{i}_1}(K_1) - 2\epsilon \\ &\geq \mathbf{W}_a^{\hat{i}_1}(\hat{p}) - 2\epsilon \\ &= \mathbf{W}_a^{\hat{i}_1}(F_{\hat{i}_1}(p, i, \hat{u}_0, \hat{y})) - 2\epsilon. \end{aligned}$$

Since ϵ and $\hat{y} \in \mathbf{R}^p$ is arbitrary, we have

$$\begin{aligned} \mathbf{W}_a^i(p) &\geq \sup_{y \in \mathbf{R}^p} \mathbf{W}_a^{\hat{i}_1}(F_{\hat{i}_1}(p, i, \hat{u}_0, y)) \\ &\geq \inf_{(j, u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \mathbf{W}_a^j(F_j(p, i, u, y)). \end{aligned}$$

□

Theorem 4.7 *Assume there exist solutions $\mathbf{W}^i, i \in \mathcal{I}$ to the dynamic programming inequalities*

$$\mathbf{W}^i(p) \geq \inf_{(j, u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \{ \mathbf{W}^j(F_j(p, i, u, y)) \} \quad (40)$$

on some nonempty domains $\text{dom} \mathbf{W}^i$ and has the property $\mathbf{W}^i(p) \geq \langle p \rangle$ for all $p \in \text{dom} \mathbf{W}^i, i \in \mathcal{I}$. Assume $\beta^i \geq 0, i \in \mathcal{I}, \beta^{i_e}(x_e) = 0$ satisfy $-\beta^i \in \text{dom} \mathbf{W}^i$ and $\mathbf{W}^i(-\beta^i) = 0$. Assume $(j^, u^*)(i, p)$ achieves the minimum in (40) for each $i \in \mathcal{I}$ and $p \in \text{dom} \mathbf{W}^i$.*

Given any $i_0 \in \mathcal{I}$, choose $p_0 = -\beta^{i_0}$ and define a measurement feedback controller $K^ \in \mathcal{K}_{mf}$ by*

$$K^*(i_0, p_0, y_{1, k}) = (i_{k+1}, u_k) = (j^*, u^*)(i_k, p_k), \quad k \geq 0 \quad (41)$$

where p_k is the information state generated according to (30) with initial condition i_0, p_0 , measurement sequence $y_{1, k}$, and control sequence $i_{0, k}, u_{0, k-1}$ obtained by (41).

Then K^ solves the measurement feedback H^∞ control problem.*

PROOF. Fix any $i_0 \in \mathcal{I}$, choose $p_0 = -\beta^{i_0}$, for any given $k \geq 0$ and measurement sequence $y_{1, k}$, we show $\mathbf{W}^i(p_l), 0 \leq l \leq k$ is non increasing, that is $\mathbf{W}^i(p_l) \geq \mathbf{W}^{i+1}(p_{l+1}), 0 \leq l \leq k-1$.

By inequality (40), we know

$$\begin{aligned}
\mathbf{W}^{i_l}(p_l) &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \{\mathbf{W}^j(F_j(p_l, i_l, u, y))\} \\
&= \sup_{y \in \mathbf{R}^p} \{\mathbf{W}^{i_{l+1}}(F_{i_{l+1}}(p_l, i_l, u_l, y))\} \\
&\geq \mathbf{W}^{i_{l+1}}(F_{i_{l+1}}(p_l, i_l, u_l, y_{l+1})) \\
&= \mathbf{W}^{i_{l+1}}(p_{l+1}).
\end{aligned} \tag{42}$$

Hence

$$0 = \mathbf{W}^{i_0}(-\beta^{i_0}) = \mathbf{W}^{i_0}(p_0) \geq \mathbf{W}^{i_k}(p_k) \geq \langle p_k \rangle, \forall k \geq 0, \forall y_{1,k}.$$

So

$$\bar{\mathbf{J}}_{-\beta^{i_0}}^{i_0} = \sup_{k \geq 0} \sup_{y_{1,k}} \{\langle p_k \rangle\} \leq 0.$$

From Proposition 4.2 and Proposition 4.5, we know the closed loop system (G, K^*) has finite gain at most γ . □

Remark 4.8 The special controller constructed from the equations (34), it is called the optimal H^∞ measurement feedback controller.

4.4 Certainty Equivalence Principle

Theorem 4.6 and Theorem 4.7 provide the necessary and sufficient conditions for the H^∞ measurement feedback control problem of switching systems. These are important results as they pointed out the exact conditions of the solvability of the measurement feedback control problem and the essential structure of the robust controllers. Unlike the full state feedback control problem, the solution of the general measurement feedback control problem essentially involves computation of the value functions \mathbf{W}^i which are functions of the information state residing in the infinite dimensional space. The involvement of infinite dimensional dynamics brings difficulty in the implementation of the controllers. In real applications, it is desirable to identify those special cases where the controllers can be computed from the finite dimensional dynamics. One such case is the certainty equivalence principle (CEP) suggested for dynamic game problems in [2]. When CEP holds, the optimal measurement feedback control strategy takes the form of the state feedback controller feeding back some estimated states. So the measurement feedback controller involves only computing the state feedback controller and the estimated states, both of which can be computed from finite dynamics. In [10], the authors identified the conditions to establish the CEP for general dynamic game problem. We follow a similar line here to develop the CEP in the context of the switching robust control problem.

Let $V^i, i \in \mathcal{I}$ are the solutions of the equations (17), K^* in (20) is the corresponding optimal H^∞ state feedback controller. Define the *minimum stress estimate* \bar{x}

$$\bar{x}(i, p) \in \operatorname{argmax}_{x \in \mathbf{R}^n} \{p(x) + V^i(x)\} \tag{43}$$

for $p \in \tilde{\mathcal{X}}, -\infty < p(x) < \infty, x \in \mathbf{R}^n$.

Theorem 4.9 Let $\mathbf{W}^i, i \in \mathcal{I}$ are solutions of (34). If for $p \in \text{dom}(\mathbf{W}^i), -\infty < p(x) < \infty, x \in \mathbf{R}^n$, it holds

$$\mathbf{W}^i(p) = (p, V^i), i \in \mathcal{I}, \quad (44)$$

then the control strategy

$$(j^*, u^*)(i, p) = K^*(\bar{x}(i, p), i) \quad (45)$$

defines the optimal H^∞ measurement feedback controller.

PROOF. According to Theorem 4.7, we only need to show that (j^*, u^*) achieves the infimum on the left hand of equations (34).

$$\begin{aligned} \mathbf{W}^i(p) &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \{ \mathbf{W}^j(F_j(p, i, u, y)) \} \\ &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \{ (F_j(p, i, u, y), V^j) \} \\ &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \sup_{\zeta \in \mathbf{R}^n} \{ F_j(p, i, u, y)(\zeta) + V^j(\zeta) \} \\ &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \sup_{\zeta \in \mathbf{R}^n} \{ \sup_{\xi \in \mathbf{R}^n} \{ p(\xi) + B_j(\xi, \zeta, i, u, y) \} + V^j(\zeta) \} \\ &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{y \in \mathbf{R}^p} \sup_{\zeta \in \mathbf{R}^n} \sup_{\xi \in \mathbf{R}^n} \left\{ p(\xi) + |g_j(\xi, u)|^2 + \rho(\xi, i, j) - \right. \\ &\quad \left. \gamma^2 |\zeta - f_j(\xi, u)|^2 - \gamma^2 |y - h_j(\xi)|^2 + V^j(\zeta) \right\} \\ &= \inf_{(j,u) \in \mathcal{I} \times U} \sup_{\xi \in \mathbf{R}^n} \{ p(\xi) + |g_j(\xi, u)|^2 + \rho(\xi, i, j) + \sup_{\zeta \in \mathbf{R}^n} \{ -\gamma^2 |\zeta - f_j(\xi, u)|^2 + V^j(\zeta) \} \}. \end{aligned} \quad (46)$$

On the other hand

$$\begin{aligned} \mathbf{W}^i(p) &= (p, V^i) \\ &= \sup_{\xi \in \mathbf{R}^n} \{ p(\xi) + V^i(\xi) \} \\ &= \sup_{\xi \in \mathbf{R}^n} \{ p(\xi) + \inf_{(j,u) \in \mathcal{I} \times U} \sup_{w \in \mathbf{R}^n} \{ |g_j(\xi, u)|^2 - \gamma^2 |w|^2 + \rho(\xi, i, j) + V^j(f_j(\xi, u) + w) \} \} \\ &= \sup_{\xi \in \mathbf{R}^n} \{ p(\xi) + \inf_{(j,u) \in \mathcal{I} \times U} \sup_{\zeta \in \mathbf{R}^n} \{ |g_j(\xi, u)|^2 - \gamma^2 |\zeta - f_j(\xi, u)|^2 + \rho(\xi, i, j) + V^j(\zeta) \} \} \\ &= \sup_{\xi \in \mathbf{R}^n} \inf_{(j,u) \in \mathcal{I} \times U} \{ p(\xi) + |g_j(\xi, u)|^2 + \rho(\xi, i, j) + \sup_{\zeta \in \mathbf{R}^n} \{ -\gamma^2 |\zeta - f_j(\xi, u)|^2 + V^j(\zeta) \} \}. \end{aligned} \quad (47)$$

So \mathbf{W}^i is a saddle value of a static game, the saddle point is $\xi^* = \bar{x}(i, p), (j^*, u^*) = K^*(\bar{x}(i, p), i)$. This shows that (j^*, u^*) achieves the infimum in (34). \square

Remark 4.10 Although it needs the value functions $\mathbf{W}^i, i \in \mathcal{I}$ to verify the conditions for CEP in Theorem 4.9, it can be seen from the proof that the condition is actually the minmax condition (46) and (47), where only the value functions of the state feedback control problems $V^i, i \in \mathcal{I}$ are involved.

5 An Example

In this section we solve, using numerical approximations, an H^∞ robust control problem for a switching system consisting of two continuous subsystems:

Subsystem 1:

$$\begin{cases} x_{k+1} = -0.9x_k + 2u_k + w_k, \\ y_k = |x_k| + v_k, \\ z_k = 0.2x_k^2 + 0.2u_k^2. \end{cases}$$

Subsystem 2:

$$\begin{cases} x_{k+1} = \sin(0.5\pi x_k)u_k + w_k, \\ y_k = \cos(x_k) + v_k, \\ z_k = 1.2(|x_k| - 2) + |u_k|. \end{cases}$$

In this example, the equilibrium point is $(x_e, i_e) = (0, 1)$.

The switching cost is taken to be

$$\rho(x, 1, 2) = \rho(x, 2, 1) = 1, \rho(x, 1, 1) = \rho(x, 2, 2) = 0, \forall x.$$

The computation is carried out in $x \in [-4, 4]$ and on the time from $k = 0$ to $k = 50$. The control takes values in $u \in [-1.5, 1.5]$. The simulation results corresponding to gain $\gamma = 1.7$ are shown below.

5.1 State Feedback Control Design

Figure 1 to Figure 9 are the simulation results based on the state feedback control design procedure from Section 3. Figure 1 depicts the state feedback value functions (15) solved from equations (17). Figure 2 to Figure 5 are the state feedback controllers computed according to (20) based on the solutions in Figure 1. By (21), we can get directly from Figure 2 and Figure 3 the switching set of the closed loop system

$$A^* = ([-4, -2.9] \cup [2.9, 4]) \times \{1\} \cup \times [-0.2, 0.2] \times \{2\}.$$

We can also see that the state feedback controllers are null initialized, that is $(j^*, u^*)(x_e, i_e) = (j^*, u^*)(0, 1) = (1, 0)$.

Figure 6 to Figure 9 simulate the closed-loop performance with a particular disturbance signal as shown in Figure 6 and initial hybrid state $(x_0, i_0) = (-3.5, 1)$. From Figure 8, we can see that there are two switches in the simulation. The first one switches to subsystem 2 at time $k = 0$, this is because the initial state $(x_0, i_0) = (-3.5, 1) \in A^*$ falls in the switching set. The second switch occurs at time $k = 21$ when the state $(x_{21}, i_{21}) = (0.1, 1)$ enters the switching set A^* again.

Though it is impossible to verify the finite gain inequality (5) completely, we have checked the inequality for different disturbance sequences and for the disturbance signal as shown in Figure 6, the inequality reads

$$\begin{aligned} \sum_{k=0}^{49} |z_k|^2 + \sum_{k=0}^{49} \rho(x_k, i_k, i_{k+1}) &= 6.5212 \\ &< 9.5172 \\ &= \gamma^2 \sum_{k=0}^{49} |w_k|^2 + \mathbf{V}^1(-3.5). \end{aligned}$$

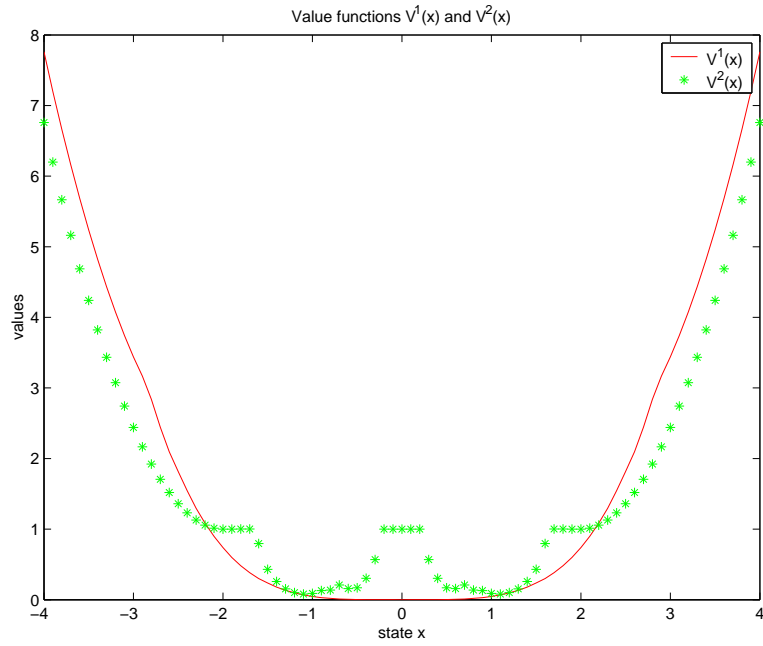


Figure 1: State feedback value functions $V^1(x)$ and $V^2(x)$

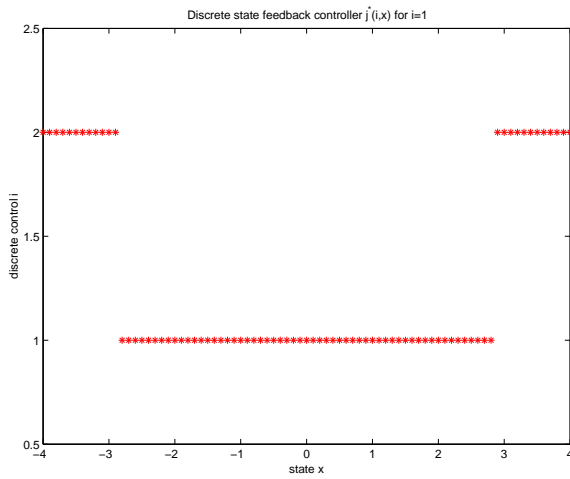


Figure 2: Discrete state feedback controller $j^*(i, x)$ for $i = 1$

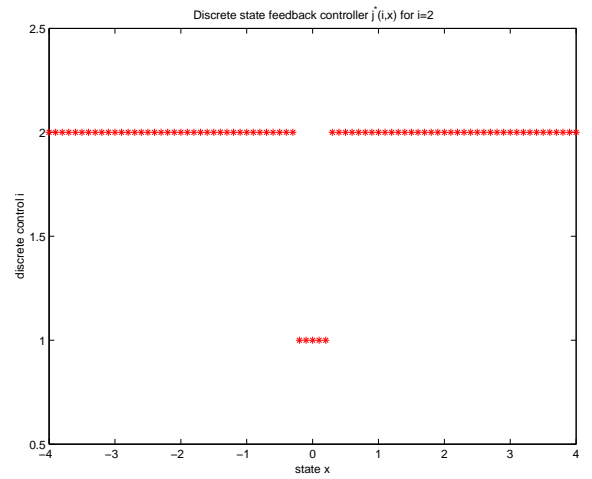


Figure 3: Discrete state feedback controller $j^*(i, x)$ for $i = 2$

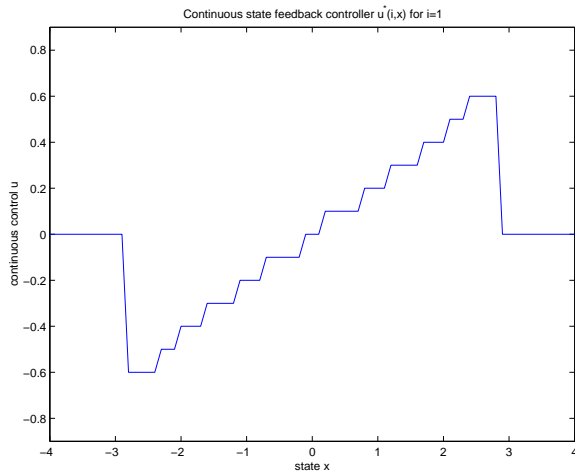


Figure 4: Continuous state feedback controller $u^*(i, x)$ for $i = 1$

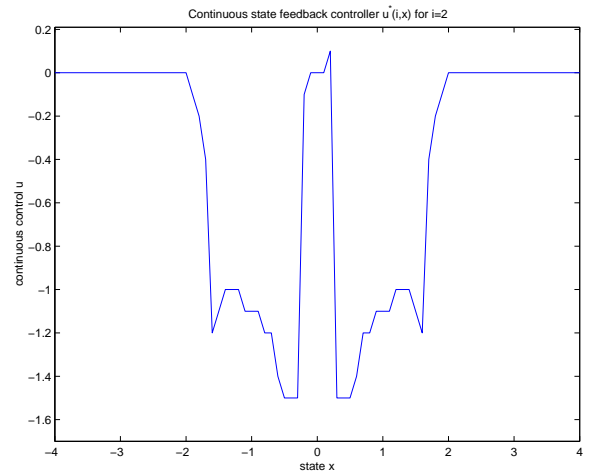


Figure 5: Continuous state feedback controller $u^*(i, x)$ for $i = 2$

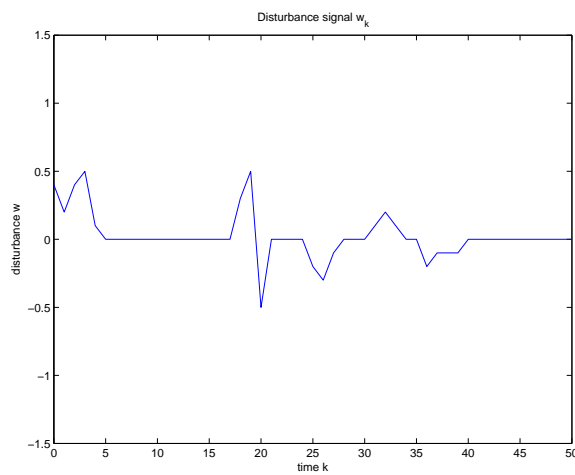


Figure 6: Disturbance signal w_k

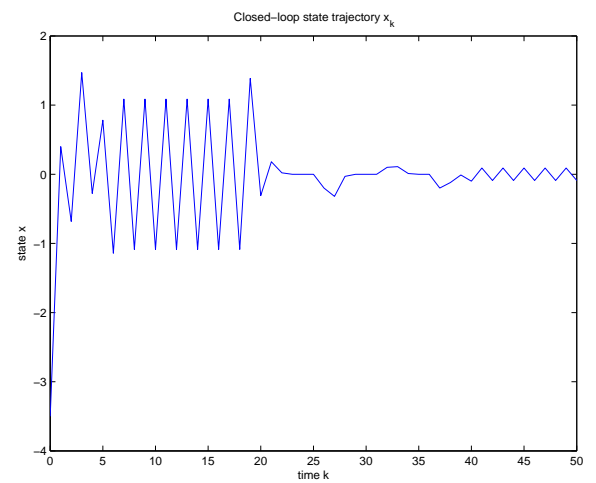


Figure 7: Closed-loop state trajectory x_k

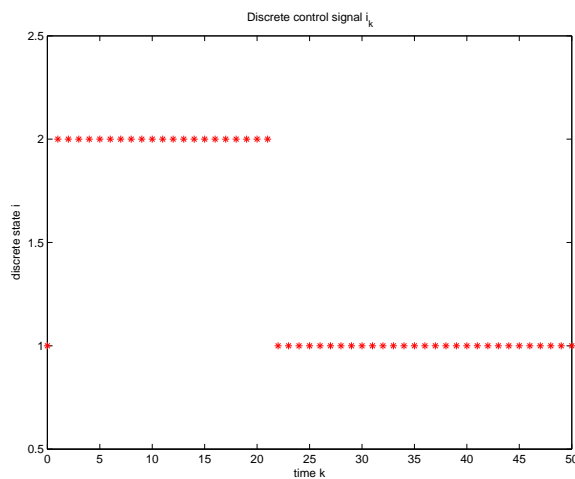


Figure 8: Discrete control signal i_k

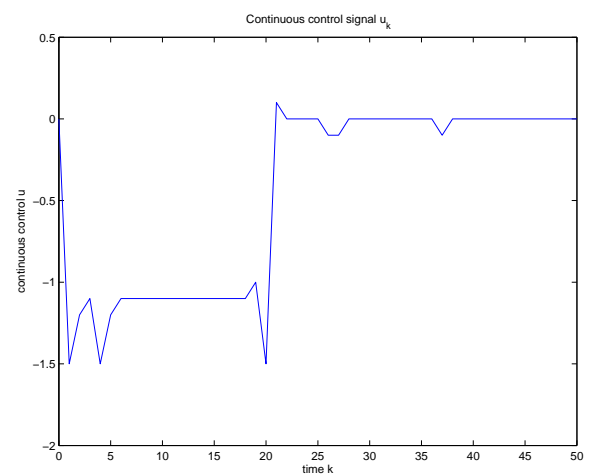


Figure 9: Continuous control signal u_k

5.2 Measurement Feedback Control Design

For measurement feedback control design simulation, besides the disturbance signal for the system dynamics in Figure 6, there is a disturbance input as in Figure 10 in the observation process. The simulation is done using the certainty equivalence principle as described in Section 4.4. Figure 11 is the disturbed observation signal. Figure 12 shows both the simulated closed loop state trajectory and the minimum stress estimated state trajectory according to (43). The control signals are depicted in Figure 13 and Figure 14. It can be clearly seen that there are two switches at time $k = 0$ and $k = 1$ respectively.

Figure 15 is the information state trajectory. The initial information state is chosen to be null, that is $p_0(x) = 0, \forall x \in \mathbf{R}^n$. Figure 16 shows the computational result concerning the minmax conditions (46) and (47) along the information state trajectory p_k which justifies the use of certainly equivalence principle in the measurement feedback control simulation.

For the closed loop system under measurement feedback controller, we have also computed that

$$\begin{aligned} \sum_{k=0}^{49} |z_k|^2 + \sum_{k=0}^{49} \rho(x_k, i_k, i_{k+1}) &= 5.2749 \\ &< 15.0160 \\ &= \gamma^2 \sum_{k=0}^{49} |w_k|^2 + \gamma^2 \sum_{k=1}^{50} |v_k|^2 + \mathbf{V}^1(-3.5). \end{aligned}$$

This verified the finite gain inequality in the case of the particular disturbances.

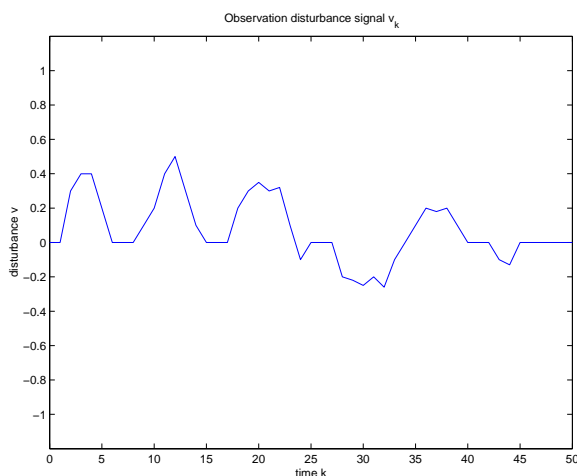


Figure 10: Observation disturbance signal v_k

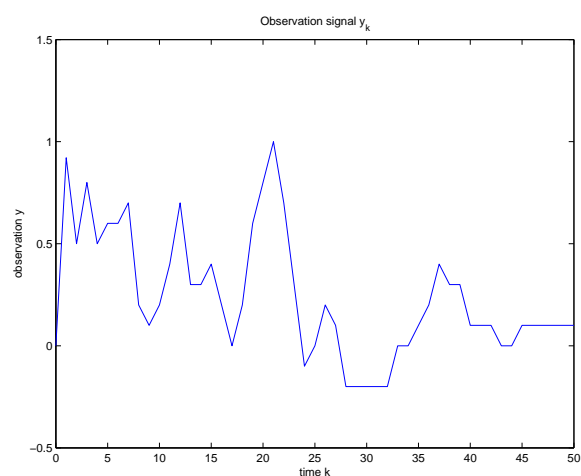


Figure 11: Observation signal y_k

6 Conclusion

This paper studied the H^∞ control problem for a class of discrete time nonlinear switching systems. The objective of the H^∞ control problem is to achieve finite l_2 gain from subsys-

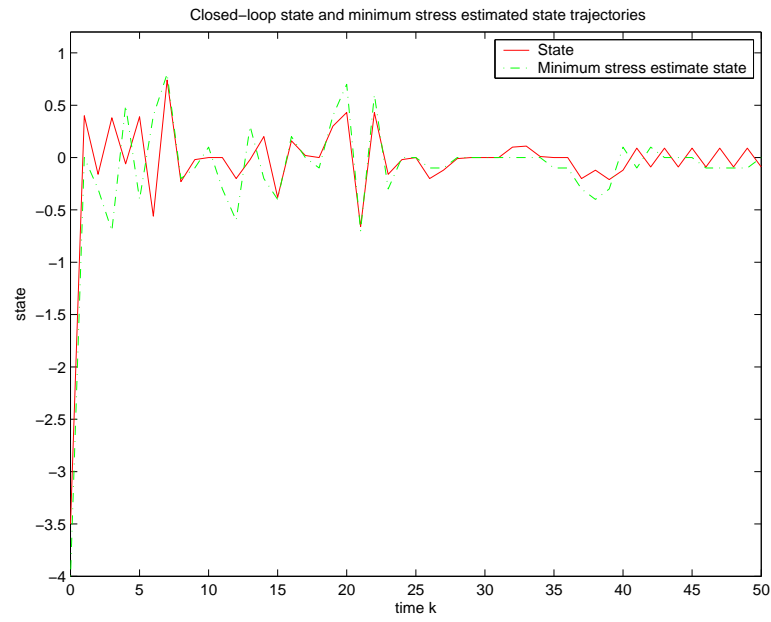


Figure 12: Simulated closed-loop state trajectory x_k and minimum stress estimated state trajectory \bar{x}_k

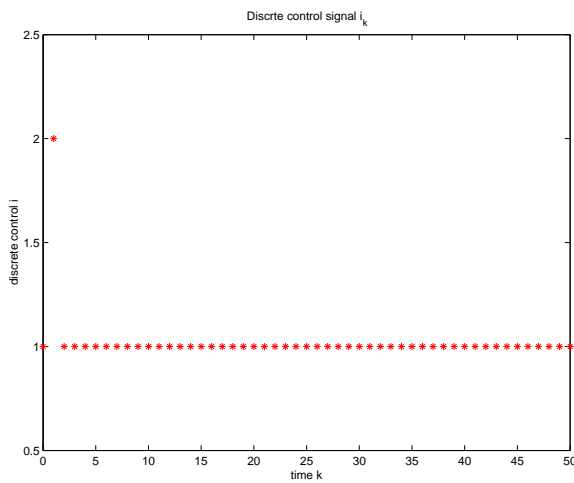


Figure 13: Discrete control signal i_k

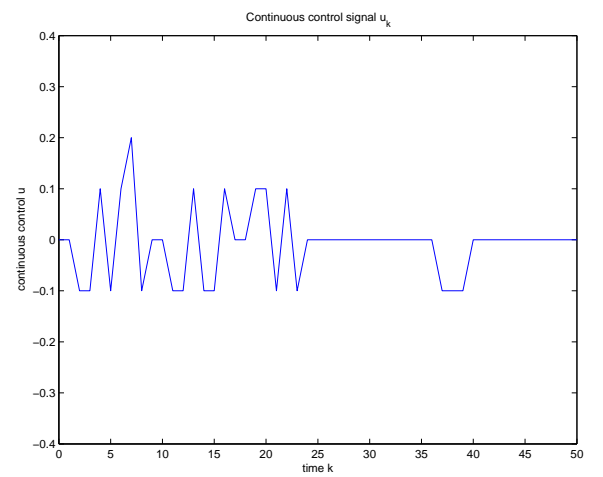


Figure 14: Continuous control signal u_k

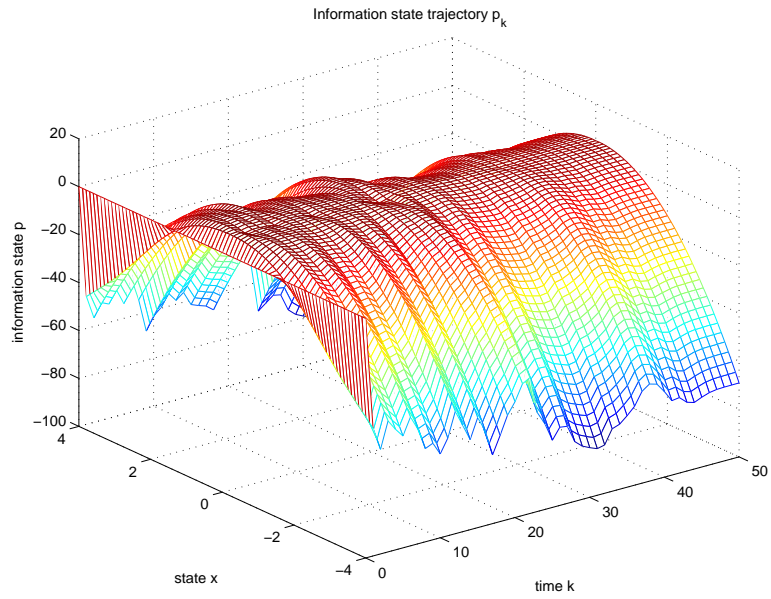


Figure 15: Information state trajectory p_k

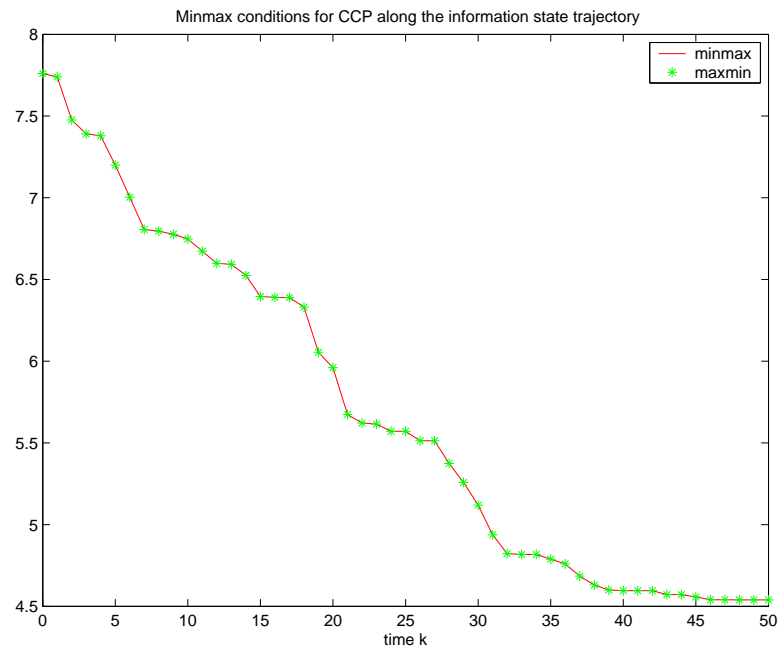


Figure 16: Verification of the CEP minmax condition along the information state trajectory

tem disturbances to the performance containing subsystem cost and switching cost. Necessary and sufficient conditions for both the state feedback control and the measurement feedback control are given in terms of dynamic programming equations (inequalities).

For the measurement feedback case, information state techniques are used. The information state employed turned out to be the state of a new infinite dimensional switching system with the measurement regarded as the new disturbances. By doing so, the original measurement feedback control problem is changed into a full state feedback control problem for a new dynamic system with information state as the new system state. Thus the problem can be solved using dynamic programming principle.

The information state can be regarded as an infinite dimensional nonlinear observers that contains all the related information contained in the measurement. In general, information state is very difficult to be solved either analytically or numerically, especially for high dimensional system. However, the optimal controllers can be computed from the finite dimensional dynamics when certainty equivalence principle holds.

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