

l^∞ -Bounded Robustness for Nonlinear Systems : Analysis and Synthesis

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Abstract

The purpose of this paper is to describe systematic analysis and design tools for robust control problems with l^∞ criteria. We first generalize the Hill-Moylan-Willems framework for dissipative systems to accommodate l^∞ criteria, and then derive state feedback and measurement feedback synthesis procedures for l^∞ robust control problems. The information state framework is used for the measurement feedback robust control problem. Necessary and sufficient conditions are proved, and new synthesis procedures using dynamic programming are presented.

Keywords: Nonlinear robust control, l^∞ criteria, dissipative systems, information states, dynamic programming, controller synthesis.

Contents

1	Introduction	3
2	Analysis	5
2.1	l^∞ Bounded Dissipation	5
2.2	Storage Functions	5
2.3	Dynamic Programming Inequality	7
2.4	Remarks Concerning Evaluation of the Available Storage	10
2.5	Mixed l^∞ -Bounded/Integral Dissipation	12
2.6	Performance and Stability	12

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3	State Feedback Synthesis	13
3.1	Problem	13
3.2	Dynamic Programming Solution	15
3.2.1	Necessity	15
3.2.2	Sufficiency	17
4	Measurement Feedback Synthesis	19
4.1	Problem Statement	19
4.2	Problem Restatement in Terms of a Cost Function	19
4.3	Equivalent Formulation Using Information States	22
4.3.1	Information States and LIB Dissipation	22
4.3.2	Recursion	23
4.3.3	Equivalent Problem	24
4.4	Dynamic Programming Solution	25
4.4.1	Necessity	26
4.4.2	Information State Controllers	29
4.4.3	Sufficiency	30
4.5	Special Case: Uniform LIB	32
5	Examples	35
5.1	Example 1 - A System with Linear Dynamics	35
5.1.1	State Feedback Synthesis	35
5.1.2	Measurement Feedback Synthesis	37
5.2	Example 2 - A System with Bilinear Dynamics	39
5.2.1	State Feedback Synthesis	39
5.2.2	Measurement Feedback Synthesis	40
A	Calculations for the Examples	42
A.1	Example 1	42
A.1.1	State Feedback	42
A.1.2	Measurement Feedback	46
A.2	Example 2	51
A.2.1	State Feedback	51
A.2.2	Measurement Feedback	56

1 Introduction

Techniques for the design of robust control systems and indeed for optimal control in general have primarily made use of integral-type performance criteria. These criteria are sometimes referred to as *soft criteria*, since a bound on the performance integral need not guarantee that an output quantity meets absolute bounds or constraints. In some applications it is important for outputs to meet hard constraints in the time domain, such as absolute regulation error always less than a specified amount. Further, persistent input signals may be present that do not have finite energy. These situations can be formulated in terms of L^∞ -type (or l^∞ -type) criteria, which might be called *hard criteria*.

Methods for analysis and design using hard criteria have been considered for some time, mostly for linear systems. We mention here a small selection of results in the literature. The l^1 optimal control problem was introduced by [23], and solutions were obtained using linear programming in [6], [7], [4], [5]. These solutions [21] can be infinite dimensional and dynamic, although finite dimensional approximation methods were developed. In [20], [21], near optimal memoryless nonlinear state feedback solutions were obtained, using controlled invariance kernels and viability theory. These results were very interesting, given that the plants were described by linear dynamics. More recently, in [8] dynamic programming equations were derived for the state feedback problem for linear systems. In [22] set-valued observers were considered, and a separation structure controller was derived for linear systems. Here, the controller was a static function of the set-valued observer state. Also, we mention the papers [9] and [17], which considered the problems of L^∞ worst-case analysis and rejection of persistent bounded disturbance for nonlinear systems.

Our objective in this paper is to describe systematic analysis and design tools for robust control problems with hard criteria. We begin by generalizing the Hill-Moylan-Willems framework for dissipative systems, originally developed for integral performance criteria, to accommodate l^∞ criteria. The generalization of the dissipation property to the l^∞ case is completely characterized in terms of a dynamic programming equation (or inequality) related to equations [9, (14) and (15)]. This is done in a way which makes use of a formal analogy between integrals and max-plus integrals (involving the (essential) supremum of a function on an interval), with links to the optimal control problems studied in [2], [3].

This analysis framework is then developed to derive state feedback and measurement feedback synthesis procedures by exploiting connections with optimal control and game theory. We prove necessary and sufficient conditions. In the state feedback case, related results are available for linear systems, e.g. [8, Section V]. For the measurement feedback l^∞ robust control problem, we employ the information state framework [14], [15], [10], and obtain dynamic controllers that feed back the information state. The information state is a generalization of observer or filter, with a state computable from measurement data. For the special case of what we refer to as the uniform l^∞ bounded dissipation problem (essentially specified in [22, Definition 4.1]), it is shown that the controllers can be chosen to feed back only a set-valued state estimate, to which the information state reduces, consistent with the separation structure of [22, Theorem 4.1]. We illustrate the synthesis procedure by applying it to simple linear and bilinear examples. Interestingly,

the certainty equivalence principle as used in linear H^∞ control [14], [1], [10] does not in general usefully apply in this l^∞ context.

This paper considers discrete time nonlinear systems for technical simplicity. We do, however, present some of the analogous continuous time equations and inequalities for comparison. The continuous time case is more technical and is considered in a separate paper.

Some notation used in this paper:

$B_0 \subset \mathbf{R}^n$	(Section 2.1, Definition 2.2)
LIB dissipative : l^∞ -bounded dissipative	(Section 2.1, Definition 2.2)
$w_{0,k-1} = \{w_0, \dots, w_{k-1}\}$	(Section 2.1, equation (2.3))
$\bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$	(Section 2.2, equation (2.5))
$\tilde{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$	(Section 4.2, equation (4.4))
$\tilde{\chi} = \{p : \mathbf{R}^n \rightarrow \tilde{\mathbf{R}}\}$	(Section 4.2, equation (4.5))
$\langle p \rangle = \sup_{x \in \mathbf{R}^n} p(x)$	(Section 4.2, equation (4.6))
support $p = \{x \in \mathbf{R}^n : p(x) > -\infty\}$	(Section 4.2, equation (4.7))
$\bar{g}(x) = \sup_{w \in \mathbf{W}} g(x, w)$ in Section 2	(Section 2.1, equation (2.2))
$\check{g}(x) = \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x, u, w)$	(Section 3.1, equation (3.2))
p_j : information state	(Section 4.3, Definition 4.5)
$F(p, u, y)$	(Section 4.3, equation (4.20))
$G(p, u, y)$	(Section 4.2, equation (4.9))
$\check{G}(p) = \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} G(p, u, y)$	(Section 4.2, equation (4.10))
$\delta_M(x) = \begin{cases} 0, & \text{if } x \in M \\ -\infty, & \text{if } x \notin M \end{cases}$	(Section 4.2, equation (4.8))
$\hat{F}(X, u, y) = F(\delta_X, u, y)$	(Section 4.5, equation (4.54))
$\hat{G}(X, u, y) = G(\delta_X, u, y)$	(Section 4.5, equation (4.55))
$\check{\hat{G}}(X) = \check{G}(\delta_X)$	(Section 4.5, equation (4.56))
V_a : available storage in Section 2	(Section 2.2, equation (2.8))
V_a : state feedback value function in Section 3	(Section 3.1, equation (3.6))
W_a : measurement feedback value function	(Section 4.4, equation (4.29))
$\text{dom}V = \{x \in \mathbf{R}^n : V(x) < +\infty\}$	(Section 2.2, equation (2.6))
$\text{dom}W = \{p \in \tilde{\chi} : -\infty < W(p) < +\infty\}$	(Section 4.4, equation(4.25))
$S \subset \text{dom}V \subset \mathbf{R}^n$	(Section 2.2, Definition 2.3)
$\tilde{S} \subset \text{dom}W \subset \tilde{\chi}$	(Section 4.4, equation (4.26))
$\bar{S} = \{\delta_X : X \subset \mathbf{R}^n\} \subset \tilde{\chi}$	(Section 4.5, equation (4.51))
$\bar{S}' = \{\text{subsets of } \mathbf{R}^n\}$	(Section 4.5, equation (4.52))
$J_p(K)$	(Section 4.2, equation (4.12))
$\bar{J}_p(K)$	(Section 4.3, equation (4.22))
$\mathcal{Y}_{0,k}(K, B)$	(Section 4.3, equation (4.23))

2 Analysis

2.1 l^∞ Bounded Dissipation

Consider nonlinear discrete-time system

$$\begin{aligned}\xi_{k+1} &= f(\xi_k, w_k) \\ z_k &= g(\xi_k, w_k)\end{aligned}\tag{2.1}$$

Here, $\xi_k \in \mathbf{R}^n$, $w_k \in \mathbf{W} \subset \mathbf{R}^s$, and $z_k \in \mathbf{R}$ are the state, disturbance input and performance output quantity, respectively.

Assumption 2.1 *Assume that*

$$\bar{g}(x) \triangleq \sup_{w \in \mathbf{W}} g(x, w) < +\infty, \quad \forall x \in \mathbf{R}^n.\tag{2.2}$$

We employ the following notation:

$$\begin{aligned}w_{0,k-1} &= \{w_0, \dots, w_{k-1}\}, \forall k \geq 0, \\ \mathcal{W}_{0,k-1} &= \{w_{0,k-1} : w_i \in \mathbf{W}, 0 \leq i \leq k-1\}, k \geq 0, \\ \mathcal{W}_{0,\infty} &= \{w_{0,\infty} : w_i \in \mathbf{W}\}.\end{aligned}\tag{2.3}$$

We adopt the convention that sets of signal sequences corresponding to the index $k = 0$ are empty, so that $\mathcal{W}_{0,-1} = \emptyset$. We also take the supremum over an empty set to equal $-\infty$.

The following definition is motivated by the disturbance rejection problem specified by [22, Definition 4.1], the worst case analysis of [9, Section IIB], the l^1 performance specification formulated in [23], [6], [7], [4], [5] and the cost functions in [2]. It is one possible definition of dissipation-like properties with l^∞ criteria. The dissipative systems framework was developed by Willems, Hill and Moylan [19], [11], [12].

Definition 2.2 *Given $B_0 \subset \mathbf{R}^n$, the system (2.1) is l^∞ -bounded (LIB) dissipative with respect to B_0 if there exists a $\beta : B_0 \rightarrow \mathbf{R}$ such that*

$$z_k \leq \beta(x_0), \quad \forall \xi_0 = x_0 \in B_0, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0.\tag{2.4}$$

2.2 Storage Functions

Denote

$$\bar{\mathbf{R}} \triangleq \mathbf{R} \cup \{+\infty\}.\tag{2.5}$$

For a function $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$, denote

$$\text{dom}V \triangleq \{x \in \mathbf{R}^n : V(x) < +\infty\}.\tag{2.6}$$

We denote $T(A) = \{f(x, w) : x \in A, w \in \mathbf{W}\}$ where $A \subset \mathbf{R}^n$. A subset $S \subset \mathbf{R}^n$ is called a T -invariant set if $T(S) \subset S$.

Definition 2.3 Let $S \subset \text{dom}V \subset \mathbf{R}^n$ be a T -invariant set. For system (2.1), $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is called a storage function on S if

$$V(x) \geq \max\left\{\max_{0 \leq i \leq k-1} g(\xi_i, w_i), V(\xi_k)\right\}, \forall x \in S, \forall w_{0,k-1} \in \mathcal{W}_{0,k-1}, \forall k \geq 0. \quad (2.7)$$

where ξ denotes the state trajectory of (2.1) with disturbance w and initial condition $\xi_0 = x$. Inequality (2.7) is called the LIB dissipation inequality.

Remark 2.4 As we will see in subsequent sections concerning synthesis, we find it useful to consider storage functions V relative to a T -invariant subsets $S \subset \text{dom}V$ (cf. [10, Chapter 4]). \square

In the general theory of dissipative systems, two particular storage functions are of special interest, viz. the available storage and the required supply. In our present context, the *available storage* $V_a : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is defined by

$$V_a(x) = \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} g(\xi_i, w_i), \quad (2.8)$$

where ξ denotes the state trajectory of (2.1) with disturbance w and initial condition $\xi_0 = x$ (this is a generalization of the usual definition of available storage [19]).

Lemma 2.5 The available storage $V_a(x)$ is a storage function on $\text{dom}V_a$ for the system (2.1). Moreover, for any storage function V on S , we have

$$S \subset \text{dom}V_a, \quad V_a(x) \leq V(x), \forall x \in S. \quad (2.9)$$

PROOF. Proof of the fact that V_a is a storage function is similar to the proofs of [19, Theorems 1 and 2]. First of all, if $x \in \text{dom}V_a$, i.e. $V_a(x) < +\infty$, then from (2.8), $\forall w \in \mathbf{W}$, $V_a(f(x, w)) \leq V_a(x) < +\infty$, i.e. $f(x, w) \in \text{dom}V_a$, so $\text{dom}V_a$ is a T -invariant set.

Now for any $x \in \mathbf{R}^n$, $k \geq 0$, $w_{0,k-1} \in \mathcal{W}_{0,k-1}$, the definition (2.8) implies

$$V_a(x) \geq \max_{0 \leq i \leq k-1} g(\xi_i, w_i). \quad (2.10)$$

Next, fix $k_1 \geq 0$, w_{0,k_1-1}^1 , $x \in \text{dom}V_a$. These determine the system (2.1) trajectory ξ_i^1 , $0 \leq i \leq k_1$. Given $\varepsilon > 0$ there exists $k_2 \geq k_1$ and w_{k_1,k_2-1}^2 such that

$$V_a(\xi_{k_1}) \leq \max_{k_1 \leq i \leq k_2-1} g(\xi_i^2, w_i^2) + \varepsilon$$

where ξ_i^2 , $k_1 \leq i \leq k_2$ is the trajectory with $\xi_{k_1}^2 = \xi_{k_1}^1$. Concatenate w_{0,k_1-1}^1 and w_{k_1,k_2-1}^2 to form an input w_{0,k_2-1} with associated concatenated trajectory ξ_i , $0 \leq i \leq k_2$. Then

$$V_a(\xi_{k_1}) \leq \max_{0 \leq i \leq k_2-1} g(\xi_i, w_i) + \varepsilon \leq V_a(x) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this inequality and (2.10) (with $k = k_1$) implies

$$V_a(x) \geq \max\left\{\max_{0 \leq i \leq k_1-1} g(\xi_i, w_i), V_a(\xi_{k_1})\right\}$$

as required.

The minimal property follows from the following observation. If V is a storage function on S as in Definition 2.3, then by (2.7) $\forall x \in S$,

$$V(x) \geq \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} g(\xi_i, w_i) = V_a(x).$$

Hence $S \subset \text{dom}V_a$. □

The following theorem shows how storage functions characterize the LIB dissipation property.

Theorem 2.6 *The system (2.1) is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$ if and only if there exists a function $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ and a T -invariant set S such that $B_0 \subset S$ and V is a storage function on S .*

PROOF. If system (2.1) is LIB dissipative with respect to B_0 , then there exists a $\beta : B_0 \rightarrow \bar{\mathbf{R}}$ such that

$$g(\xi_k, w_k) = z_k \leq \beta(x_0), \quad \forall x_0 \in B_0, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0,$$

hence from (2.8) we have $V_a(x_0) \leq \beta(x_0)$ for all $x_0 \in B_0$. Thus $B_0 \subset \text{dom}V_a$, and from Lemma 2.5, V_a is a storage function on $\text{dom}V_a$ with $B_0 \subset \text{dom}V_a$, as required.

Conversely, if V is a storage function on S , then

$$z_k = g(\xi_k, w_k) \leq \max_{0 \leq i \leq k} g(\xi_i, w_i) \leq V(x_0) < +\infty, \quad \forall x_0 \in S, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0,$$

Since $B_0 \subset S$, $\beta(x_0) \triangleq V(x_0)$ satisfies (2.4). □

Corollary 2.7 *System (2.1) is LIB dissipative with respect to B_0 if and only if $B_0 \subset \text{dom}V_a$.*

Remark 2.8 In fact, if there exists a storage function V on a T -invariant set S , then the system (2.1) is LIB dissipative with respect to S . Moreover, $\text{dom}V_a$ is the largest set on which the system possesses the LIB dissipative property. □

2.3 Dynamic Programming Inequality

We now give an “infinitesimal”, or, precisely, a one-step, dynamic programming inequality that characterizes storage functions, and hence by Theorem 2.6 the LIB dissipation property.

For $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ and $S \subset \text{dom}V \subset \mathbf{R}^n$, the *LIB dynamic programming equation* is

$$V(x) = \sup_{w \in \mathbf{W}} \max\{g(x, w), V(f(x, w))\}, \forall x \in S, \quad (2.11)$$

or equivalently

$$V(x) = \max\{\bar{g}(x), \sup_{w \in \mathbf{W}} V(f(x, w))\}, \forall x \in S, \quad (2.12)$$

where $\bar{g}(x)$ is defined in (2.2). The analogous *LIB dynamic programming inequality* is

$$V(x) \geq \sup_{w \in \mathbf{W}} \max\{g(x, w), V(f(x, w))\}, \forall x \in S. \quad (2.13)$$

Remark 2.9 The analogous LIB dynamic programming equation for the continuous time system $\dot{\xi} = f(\xi, w)$, $z = g(\xi, w)$, is the partial differential equation [13] (see also [2], [3], [9]):

$$\sup_{w \in \mathbf{W}} \max\{g(x, w) - V(x), \nabla V(x)f(x, w)\} = 0 \quad \text{in } S \subset \text{dom}V. \quad (2.14)$$

□

Theorem 2.10 *Given a function $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ and a T -invariant set S , V is a storage function on S if and only if (V, S) is a solution of (2.13).*

PROOF. Assume that V is a storage function satisfying (2.7). Setting $k = 1$ and $w_0 = w \in \mathbf{W}$ this inequality implies

$$V(x) \geq \max\{g(\xi_0, w), V(\xi_1)\}, \quad \forall x \in S$$

which is simply

$$V(x) \geq \max\{g(x, w), V(f(x, w))\}, \quad \forall x \in S.$$

Since this holds for all $w \in \mathbf{W}$ we obtain (2.13).

Conversely, assume (V, S) satisfying (2.13). Let $k \geq 0$ and select $w_{0,k-1} \in \mathcal{W}_{0,k-1}$ and $x_0 \in S$. This determines a trajectory ξ_i , $0 \leq i \leq k$. Since S is a T -invariant set, iterating (2.13) we find that

$$V(x_0) \geq V(\xi_i), \quad \forall 0 \leq i \leq k; \quad V(\xi_i) \geq g(\xi_i, w_i), \quad \forall 0 \leq i \leq k-1.$$

Therefore $V(x_0) \geq V(\xi_k)$ and $V(x_0) \geq \max_{0 \leq i \leq k-1} g(\xi_i, w_i)$. This implies (2.7). □

Theorem 2.11 *(Necessity) If system (2.1) is LIB dissipative with respect to B_0 , then the available storage V_a defined by (2.8) satisfies*

- (i) $B_0 \subset \text{dom}V_a$;
- (ii) $V_a(x) \geq \bar{g}(x)$, $\forall x \in \mathbf{R}^n$, where $\bar{g}(x)$ is defined in (2.2).

(iii) $\text{dom}V_a$ is a T -invariant set and the dynamic programming relation holds:

$$V_a(x) = \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max\left\{ \max_{0 \leq i \leq j-1} g(\xi_i, w_i), V_a(\xi_j) \right\}, \quad \forall x \in \text{dom}V_a, \forall j \geq 0. \quad (2.15)$$

i.e. V_a solve the dynamic programming equation (2.11) with $S = \text{dom}V_a$.

(iv) If (V, S) satisfies the LIB dynamic programming inequality (2.13), then

$$S \subset \text{dom}V_a, \quad V_a(x) \leq V(x), \forall x \in S.$$

PROOF. (i) and (ii) are obvious by the definition of V_a . For (iii), from Lemma 2.5, we only need to prove

$$V_a(x) \leq \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max\left\{ \max_{0 \leq i \leq j-1} g(\xi_i, w_i), V_a(\xi_j) \right\}, \quad \forall x \in \text{dom}V_a, \forall j \geq 0. \quad (2.16)$$

When $j = 0$, the inequality holds obviously. Now fix $j > 0$ and fix $x \in \text{dom}V_a$, $\varepsilon > 0$. Then there exists $k \geq 0$ such that

$$V_a(x) \leq \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} g(\xi_i, w_i) + \varepsilon$$

If $k \leq j$, we have

$$\begin{aligned} V_a(x) &\leq \sup_{w_{0,j-1}} \max_{0 \leq i \leq j-1} g(\xi_i, w_i) + \varepsilon \\ &\leq \sup_{w_{0,j-1}} \max\left\{ \max_{0 \leq i \leq j-1} g(\xi_i, w_i), V_a(\xi_j) \right\} + \varepsilon \end{aligned}$$

If $j < k$,

$$\begin{aligned} V_a(x) &\leq \sup_{w_{0,j-1}} \sup_{w_{j,k-1}} \max\left\{ \max_{0 \leq i \leq j-1} g(\xi_i, w_i), \max_{j \leq i \leq k-1} g(\xi_i, w_i) \right\} + \varepsilon \\ &\leq \sup_{w_{0,j-1}} \max\left\{ \max_{0 \leq i \leq j-1} g(\xi_i, w_i), V_a(\xi_j) \right\} + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have proven (2.16).

(iv) follows from Lemma 2.5 and Theorem 2.10. \square

Theorem 2.12 (Sufficiency) *If there exists $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ and T -invariant set S satisfying the LIB dynamic programming inequality (2.13), then for system (2.1),*

$$z_k \leq V(x_0), \quad \forall \xi_0 = x_0 \in S, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0.$$

Moreover, if $B_0 \subset S$, then system (2.1) is LIB dissipative with respect to B_0 .

PROOF. The conclusion comes directly from Theorem 2.6 and Theorem 2.10. \square

2.4 Remarks Concerning Evaluation of the Available Storage

Evaluation or computation of the storage function V_a defined by (2.8) is an important problem, considered at length in [9]. In this subsection we consider different ways in which this can be done. We begin with proving an equivalent definition of the available storage.

Lemma 2.13 *We have the following identity for the available storage:*

$$V_a(x) = \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} g(\xi_{k-1}, w_{k-1}) \quad (2.17)$$

where ξ denotes the state trajectory of (2.1) with disturbance w and initial condition $\xi_0 = x$.

PROOF. Since $\max_{0 \leq i \leq k-1} g(\xi_i, w_i) \geq g(\xi_{k-1}, w_{k-1})$, $V_a(x) \geq \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} g(\xi_{k-1}, w_{k-1})$.

Now we will prove that

$$V_a(x) \leq \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} g(\xi_{k-1}, w_{k-1}), \quad \forall x \in \text{dom} V_a. \quad (2.18)$$

Fix $x \in \text{dom} V_a$, $\forall \varepsilon > 0$, there exists $k_0 \geq 0$, $w_{0,k_0-1}^0 \in \mathcal{W}_{0,k_0-1}$ such that

$$V_a(x) \leq \max_{0 \leq i \leq k_0-1} g(\xi_i^0, w_i^0) + \varepsilon.$$

Since k_0 is a finite number, there exists $1 \leq i_0 \leq k_0$ such that

$$\max_{0 \leq i \leq k_0-1} g(\xi_i^0, w_i^0) = g(\xi_{i_0-1}^0, w_{i_0-1}^0).$$

Hence

$$\begin{aligned} V_a(x) &\leq g(\xi_{i_0-1}^0, w_{i_0-1}^0) + \varepsilon \\ &\leq \sup_{w_{0,i_0-1} \in \mathcal{W}_{0,i_0-1}} g(\xi_{i_0-1}, w_{i_0-1}) + \varepsilon \\ &\leq \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} g(\xi_{k-1}, w_{k-1}) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have proved (2.18). \square

Using Lemma 2.13, we have

$$\begin{aligned} V_a(x) &= \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} \bar{g}(\xi_i) \\ &= \sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} g(\xi_k, w_k) \\ &= \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \bar{g}(\xi_k), \quad \forall x \in \mathbf{R}^n, \end{aligned} \quad (2.19)$$

where $\bar{g}(x)$ is defined in (2.2). This identity suggests two iterative approaches for evaluating V_a .

Define

$$V_{a1}(x, k) \triangleq \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \max_{0 \leq i \leq k-1} \bar{g}(\xi_i) \quad (2.20)$$

and

$$V_{a2}(x, k) \triangleq \sup_{w_0, k-1 \in \mathcal{W}_{0, k-1}} \bar{g}(\xi_k). \quad (2.21)$$

where $\bar{g}(x)$ is defined in (2.2).

Then we have

$$\begin{aligned} V_a(x) &= \sup_{k \geq 0} V_{a1}(x, k) = \lim_{k \rightarrow \infty} V_{a1}(x, k) \\ &= \sup_{k \geq 0} V_{a2}(x, k) \end{aligned} \quad (2.22)$$

since $k \mapsto V_{a1}(x, k)$ is monotone non-decreasing. However, the dynamic programming recursions for these two functions are quite different. The function $V_{a1}(x, k)$ solves the dynamic programming recursion

$$\begin{aligned} V_{a1}(x, k) &= \max\{\bar{g}(x), \sup_{w \in \mathbf{W}} V_{a1}(f(x, w), k-1)\} \\ V_{a1}(x, 1) &= \bar{g}(x) \end{aligned} \quad (2.23)$$

while $V_{a2}(x, k)$ solves the (simpler) dynamic programming recursion

$$\begin{aligned} V_{a2}(x, k) &= \sup_{w \in \mathbf{W}} V_{a2}(f(x, w), k-1) \\ V_{a2}(x, 0) &= \bar{g}(x) \end{aligned} \quad (2.24)$$

Then

$$V_{a1}(x, k+1) = \max\{V_{a2}(x, k), V_{a1}(x, k)\}. \quad (2.25)$$

Another derivation for the dynamic programming equality (2.15) satisfied by V_a can be obtained by considering sets of possible states, as follows. Denote

$$X(x_0) = \{x \in \mathbf{R}^n : \exists k \geq 0, \exists w_0, k-1 \in \mathcal{W}_{0, k-1} \text{ such that } \xi_0 = x_0, \xi_k = x\},$$

then

$$V_a(x_0) = \sup_{x \in X(x_0)} \bar{g}(x)$$

Since

$$X(x_0) = \{x_0\} \bigcup \left\{ \bigcup_{w_0 \in \mathbf{W}} X(f(x_0, w_0)) \right\}$$

we have

$$V_a(x_0) = \max\{\bar{g}(x_0), \sup_{w_0 \in \mathbf{W}} V_a(f(x_0, w_0))\},$$

which is the LIB dynamic programming equality (2.12).

Remark 2.14 Invariant kernels play an important role in the papers [21], [22]. The LIB dissipation property is equivalent to the largest invariant set under the uncertain dynamics being contained in a set of the form $D = \{x : \bar{g}(x) \leq \beta\}$. The set $X(x_0)$ we defined above is the smallest T -invariant set which contains $\{x_0\}$. Also $X(x_0) \subset D$ whenever the system is LIB dissipative with respect to B_0 and $x_0 \in B_0$. \square

2.5 Mixed l^∞ -Bounded/Integral Dissipation

In some applications it might be desired to incorporate an integral (sum) into the performance criteria, and obtain mixed l^∞ -bounded/integral criteria. Suppose we have a system

$$\begin{aligned}\xi_{k+1} &= f(\xi_k, w_k) \\ z_{1k} &= g_1(\xi_k, w_k) \\ z_{2k} &= g_2(\xi_k, w_k)\end{aligned}\tag{2.26}$$

with disturbance input w and real-valued performance output quantities z_1, z_2 . The system (2.26) is *l^∞ -bounded/integral (LIB/I) dissipative* if there exists a finite real-valued bias function β such that

$$\sum_{i=0}^{k-1} z_{1i} + z_{2k} \leq \beta(x_0), \quad \forall x_0 \in \mathbf{R}^n, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0\tag{2.27}$$

where the performance quantities are evaluated along the corresponding trajectory of (2.26) with initial condition $\xi(0) = x_0$.

The definition of LIB/I dissipation via inequality (2.27) consists of an integral term (involving z_1) and a point evaluation term (involving z_2). The z_1 term gives a *soft* constraint, while the z_2 term gives a *hard* constraint. However, it is not actually more general than LIB dissipation. To see this, a problem specified by (2.26) and (2.27) is equivalent to an augmented LIB dissipation problem given by

$$\begin{aligned}\tilde{\xi}_{k+1} &= \tilde{f}(\tilde{\xi}_k, w_k) \\ \tilde{z}_k &= \tilde{g}(\tilde{\xi}_k, w_k)\end{aligned}\tag{2.28}$$

and

$$\tilde{z}_k \leq \tilde{\beta}(\tilde{x}_0), \quad \forall x_0 \in \mathbf{R}^n, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0\tag{2.29}$$

where

$$\begin{aligned}\tilde{\xi}_k &= \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \eta_{k+1} = \eta_k + g_1(\xi_k, w_k), \tilde{x}_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ \tilde{f}(\tilde{\xi}, w) &= \begin{pmatrix} f(\xi, w) \\ \eta + g_1(\xi, w) \end{pmatrix}, \tilde{g}(\tilde{\xi}, w) = g_2(\xi, w) + \eta.\end{aligned}\tag{2.30}$$

2.6 Performance and Stability

The notion of LIB dissipation abstracts the approach to worst case analysis in [9]; the definition (2.8) of available storage corresponds to the function defined by equations (2) and (3) in [9]. Storage functions for LIB dissipative systems can be used to analyze L^∞ gain functions and induced L^∞ gains over bounded signals [9].

In many applications asymptotic stability to an equilibrium, or stability about an equilibrium, is an issue. We say that the system (2.1) has an *equilibrium* at $x = 0$ if $0 \in \mathbf{W}$, $f(0, 0) = 0$, $g(0, 0) = 0$. The next theorem is an example of a stability theorem for LIB dissipative systems.

Theorem 2.15 *Let V be a storage function on $S \subset \text{dom}V$ for an LIB dissipative system (2.1) with equilibrium $x = 0$, where $g(x, w) = |x|$.*

(i) *If $x_0 \in S$, then the state is bounded as follows:*

$$|\xi_k| \leq V(\xi_k) \leq V(x_0), \quad \forall k \geq 0 \quad (2.31)$$

whenever $w_{0,\infty} \in \mathcal{W}_{0,\infty}$.

(ii) *Assume in addition that V is continuous and $V(0) = 0$. Then for any $w_{0,\infty} \in \mathcal{W}_{0,\infty}$ the system (2.1) is stable, and V is a positive definite Lyapunov function.*

(iii) *Assume in addition that $V(f(x, 0)) = V(x) \Rightarrow x = 0$, and that $f(x, 0)$ is continuous. Then the system (2.1) is asymptotically stable when $w = 0$.*

PROOF. Since V is a storage function on S , from Definition 2.3, when $\xi_0 = x_0 \in S$, $\forall k \geq 0, \xi_k \in S$ and

$$|\xi_k| = g(\xi_k, w) \leq V(\xi_k); \quad V(\xi_k) \leq V(x_0).$$

This immediately implies (2.31).

If also V is continuous and $V(0) = 0$, then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0| < \delta$ implies $V(x_0) < \varepsilon$. Then (2.31) implies $|\xi_k| < \varepsilon$ for all $k \geq 0$.

Since V is positive definite and $V(\xi_{k+1}) \leq V(\xi_k), \forall k \geq 0$, V is a Lyapunov function.

When $V(f(x, 0)) = V(x) \Rightarrow x = 0$, $f(x, 0)$ is continuous and $w = 0$, since ξ_k is bounded, from the Invariance Principle [16, Chapter 1, Theorem 6.3], $\xi_k \rightarrow 0$ as $k \rightarrow \infty$. So the system is asymptotically stable when $w = 0$. \square

A useful application of LIB dissipation is to bound the maximum fluctuation of a regulated variable $r = c(x)$ from its set-point r_0 by using the performance output $g(x, w) = |c(x) - r_0|$:

$$V(\xi_k) \geq |c(\xi_k) - r_0| = |r_k - r_0|, \quad \forall k \geq 0, \quad (2.32)$$

whenever $w_{0,\infty} \in \mathcal{W}_{0,\infty}$.

3 State Feedback Synthesis

With the tools developed in the previous section in hand, we turn to the problem of finding state feedback controllers achieving LIB dissipation for the closed loop.

3.1 Problem

Consider nonlinear discrete-time system

$$\begin{aligned} \xi_{k+1} &= f(\xi_k, u_k, w_k) \\ z_k &= g(\xi_k, u_k, w_k) \end{aligned} \quad (3.1)$$

Here, $\xi_k \in \mathbf{R}^n, u_k \in \mathbf{U} \subset \mathbf{R}^m, w_k \in \mathbf{W} \subset \mathbf{R}^s$ and $z_k \in \mathbf{R}$ are the state, control input, disturbance and performance output quantity, respectively.

Assumption 3.1 Assume that

$$\check{g}(x) \triangleq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x, u, w) > -\infty, \quad \forall x \in \mathbf{R}^n. \quad (3.2)$$

Besides (2.3), we employ the following notation:

$$\begin{aligned} x_{0,k} &= \{x_0, \dots, x_k\}, \forall k \geq 0, \\ u_{0,k} &= \{u_0, \dots, u_k\}, \forall k \geq 0, \\ \mathcal{X}_{0,k} &= \{x_{0,k} : x_i \in \mathbf{R}^n, 0 \leq i \leq k\}, \\ \mathcal{U}_{0,k} &= \{u_{0,k} : u_i \in \mathbf{U}, 0 \leq i \leq k\}, \\ \mathcal{X}_{0,\infty} &= \{x_{0,\infty} : x_i \in \mathbf{R}^n\}, \\ \mathcal{U}_{0,\infty} &= \{u_{0,\infty} : u_i \in \mathbf{U}\}. \end{aligned} \quad (3.3)$$

An *admissible* state feedback controller is a causal map $K : \mathcal{X}_{0,\infty} \rightarrow \mathcal{U}_{0,\infty}$, meaning that for each time $k \geq 0$ if $x^1, x^2 \in \mathcal{X}_{0,\infty}$ and $x_l^1 = x_l^2$ for all $0 \leq l \leq k$ then $K(x^1)_k = K(x^2)_k$, i.e., the control at time k is independent of future states. Denote by \mathcal{K}_{state} the class of such admissible state feedback controllers. We sometimes abuse notation by writing $u_k = K(x_{0,k})$ or $u = K(x)$.

Problem: Given $B_0 \subset \mathbf{R}^n$, find a state feedback controller $K \in \mathcal{K}_{state}$ such that the closed-loop system is LIB dissipative with respect to B_0 .

When using state feedback controller $K \in \mathcal{K}_{state}$, the closed-loop system is

$$\begin{aligned} \xi_{k+1} &= f(\xi_k, K(\xi_{0,k}), w_k) \\ z_k &= g(\xi_k, K(\xi_{0,k}), w_k) \end{aligned} \quad (3.4)$$

For this closed-loop system, we define

$$\beta_a^K(x) \triangleq \sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{g(\xi_k, u_k, w_k) : u = K(\xi)\}, \quad \forall x \in \mathbf{R}^n \quad (3.5)$$

where ξ denotes the state trajectory of (3.4) with disturbance w and initial condition $\xi_0 = x$. In fact, $\beta_a^K(x)$ is the available storage of the closed-loop system (with controller K) (see (2.19)).

Further define the *state feedback value function*

$$V_a(x) = \inf_{K \in \mathcal{K}_{state}} \beta_a^K(x), \quad \forall x \in \mathbf{R}^n. \quad (3.6)$$

By Assumption 3.1,

$$V_a(x) \geq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x, u, w) = \check{g}(x) > -\infty, \quad \forall x \in \mathbf{R}^n.$$

i.e. $V_a : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$.

3.2 Dynamic Programming Solution

In Section 2.3 we saw the importance of the LIB dynamic programming inequality (2.13) and equation (2.11) in characterizing LIB dissipation. A similar inequality and equation arises when dynamic programming techniques are applied to the minimax game specified by (3.6). As we shall see, it will be useful to consider the dynamic programming equation or inequality as holding on a subset of the domain of the solution function V .

Let $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ and $S \subset \text{dom}V \subset \mathbf{R}^n$. The *state feedback dynamic programming equation* is

$$V(x) = \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} \max\{g(x, u, w), V(f(x, u, w))\}, \forall x \in S, \quad (3.7)$$

The analogous *state feedback dynamic programming inequality* is

$$V(x) \geq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} \max\{g(x, u, w), V(f(x, u, w))\}, \forall x \in S. \quad (3.8)$$

Remark 3.2 The analogous partial differential equation for the continuous time system $\dot{\xi} = f(\xi, u, w)$, $z = g(\xi, u, w)$, is

$$\inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} \max\{g(x, u, w) - V(x), \nabla V(x) f(x, u, w)\} = 0 \quad \text{in } S \subset \text{dom}V. \quad (3.9)$$

□

3.2.1 Necessity

Theorem 3.3 *If there exists a state feedback controller $K_0 \in \mathcal{K}_{state}$ such that system (3.4) (with $K = K_0$) is LIB dissipative with respect to B_0 , then the function $V_a : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ defined by (3.6) satisfies:*

- (i) $B_0 \subset \text{dom}V_a$;
- (ii) $V_a(x) \geq \check{g}(x)$, $\forall x \in \mathbf{R}^n$, where $\check{g}(x)$ is defined in (3.2);
- (iii) If $\beta_a^{K_0}(x)$ is finite, then $\forall j \geq 0, \forall w_{0,j-1} \in \mathcal{W}_{0,j-1}$,

$$\beta_a^{K_0}(x) \geq \max\left\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\right\}, \quad \xi_0 = x, u_i = K_0(\xi_{0,i}) \quad (3.10)$$

- (iv) If K_ε is an ε -optimal controller (i.e. $\beta_a^{K_\varepsilon}(x) \leq V_a(x) + \varepsilon$), then $\forall j \geq 0, \forall w_{0,j-1} \in \mathcal{W}_{0,j-1}$,

$$V_a(x) + \varepsilon \geq \max\left\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\right\}, \quad \xi_0 = x, u_i = K_\varepsilon(\xi_{0,i}). \quad (3.11)$$

- (v) *The dynamic programming relation holds*

$$V_a(x) = \inf_{K \in \mathcal{K}_{state}} \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \left\{ \max\left\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\right\} : \xi_0 = x, u_i = K(\xi_{0,i}) \right\}, \quad (3.12)$$

$\forall x \in \text{dom}V_a, \forall j \geq 0,$

i.e. V_a solves the dynamic programming equation (3.7), with $S = \text{dom}V_a$.

PROOF.

(i) From Corollary 2.7, $B_0 \subset \text{dom}\beta_a^{K_0}$, hence $B_0 \subset \bigcup_{K \in \mathcal{K}_{state}} \text{dom}\beta_a^K = \text{dom}V_a$.

(ii) From (ii) of Theorem 2.11, $\forall K \in \mathcal{K}_{state}, \forall x \in \mathbf{R}^n$,

$$\beta_a^K(x) \geq \sup_{w \in \mathbf{W}} g(x, K(x), w) \geq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x, u, w) = \check{g}(x),$$

hence

$$V_a(x) \geq \check{g}(x), \forall x \in \mathbf{R}^n.$$

(iii) If $j = 0$, $\beta_a^{K_0}(x) \geq V_a(x) = V_a(\xi_0)$ and (3.10) holds.

Now fix $j > 0$ and fix $w_{0,j-1} \in \mathcal{W}_{0,j-1}$, then $\xi_{0,j}$ is fixed, for any $k \geq 0$ and any $\tilde{w}_{0,k}$, denote

$$\hat{w}_i = \begin{cases} w_i, & 0 \leq i \leq j-1 \\ \tilde{w}_{i-j}, & i \geq j \end{cases} \quad (3.13)$$

and

$$K_0^j(\xi_{j,j+i}) = K_0(\xi_{0,j+i}), \quad i \geq 0. \quad (3.14)$$

Then we can prove that

$$\begin{aligned} \beta_a^{K_0}(x) &= \sup_{i \geq 0} \sup_{\bar{w}_{0,i}} \{g(\xi_i, u_i, \bar{w}_i) : u = K_0(\xi)\} \\ &\geq \sup_{i \geq 0} \sup_{\hat{w}_{0,i}} \{g(\xi_i, u_i, \hat{w}_i) : u = K_0(\xi)\} \\ &= \max \left\{ \max_{0 \leq i \leq j-1} \{g(\xi_i, u_i, w_i) : u = K_0(\xi)\}, \sup_{i \geq j} \sup_{\hat{w}_{0,i}} \{g(\xi_i, u_i, \hat{w}_i) : u = K_0(\xi)\} \right\} \\ &= \max \left\{ \max_{0 \leq i \leq j-1} \{g(\xi_i, u_i, w_i) : u = K_0(\xi)\}, \right. \\ &\quad \left. \sup_{k \geq 0} \sup_{\tilde{w}_{0,k}} \{g(\tilde{\xi}_k, u_k, \tilde{w}_k) : \tilde{\xi}_0 = \xi_j, u = K_0^j(\tilde{\xi})\} \right\} \\ &= \max \left\{ \max_{0 \leq i \leq j-1} \{g(\xi_i, u_i, w_i) : u = K_0(\xi)\}, \beta_a^{K_0^j}(\xi_j) \right\} \\ &\geq \max \left\{ \max_{0 \leq i \leq j-1} \{g(\xi_i, u_i, w_i) : u = K_0(\xi)\}, V_a(\xi_j) \right\} \\ &= \max \left\{ \max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j) \right\} : u = K_0(\xi). \end{aligned}$$

(iv) Apply part (iii) with $K_0 = K_\varepsilon$.

(v) $\forall x \in \text{dom}V_a, \forall \varepsilon > 0, \exists K_\varepsilon \in \mathcal{K}_{state}$, such that

$$V_a(x) \geq \beta_a^{K_\varepsilon}(x) - \varepsilon.$$

Hence from (iv), $\forall j \geq 0$, we have

$$\begin{aligned} V_a(x) &\geq \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max \left\{ \max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j) \right\} : u = K_\varepsilon(\xi) - \varepsilon \\ &\geq \inf_{K \in \mathcal{K}_{state}} \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max \left\{ \max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j) \right\} - \varepsilon \end{aligned}$$

Since ε is arbitrary, we have

$$V_a(x) \geq \inf_{K \in \mathcal{K}_{state}} \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max \left\{ \max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j) \right\}. \quad (3.15)$$

To prove the opposite inequality, define

$$U(x) = \inf_{K \in \mathcal{K}_{state}} \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \max\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\}, \forall x \in \text{dom}V_a. \quad (3.16)$$

We have from (3.15), $U(x) \leq V_a(x) < +\infty$, moreover, $U(x) \geq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x, u, w) = \check{g}(x) > -\infty$, so $U(x)$ is finite.

$\forall \varepsilon > 0, \exists K^1 \in \mathcal{K}_{state}$, such that

$$U(x) \geq \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \{\max\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\} : u = K^1(\xi)\} - \varepsilon.$$

$\forall \xi \in \text{dom}V_a, \exists K_\xi^2 \in \mathcal{K}_{state}$, such that

$$V_a(\xi) \geq \beta_a^{K_\xi^2}(\xi) - \varepsilon.$$

Now define $K^3 \in \mathcal{K}_{state}$ by

$$K^3(\xi_{0,i}) = \begin{cases} K^1(\xi_{0,i}), & 0 \leq i \leq j-1 \\ K_{\xi_j}^2(\xi_{j,i}), & i \geq j \end{cases} \quad (3.17)$$

then $\forall k \geq j, \forall w_{0,k} \in \mathcal{W}_{0,k}$, we have

$$\begin{aligned} U(x) &\geq \{\max\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), V_a(\xi_j)\} : u = K^1(\xi)\} - \varepsilon \\ &\geq \{\max\{\max_{0 \leq i \leq j-1} g(\xi_i, u_i, w_i), \beta_a^{K_{\xi_j}^2}(\xi_j)\} : u = K^1(\xi)\} - 2\varepsilon \\ &\geq \sup_{0 \leq i \leq k} \{g(\xi_i, u_i, w_i) : u = K^3(\xi)\} - 2\varepsilon \end{aligned}$$

so we have

$$U(x) \geq \beta_a^{K^3}(x) - 2\varepsilon \geq V_a(x) - 2\varepsilon.$$

Since ε is arbitrary, we have

$$V_a(x) \leq U(x). \quad (3.18)$$

Finally from (3.15) and (3.18), the dynamic programming equation (3.12) holds. \square

3.2.2 Sufficiency

Definition 3.4 Given a function $V : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ and a nonempty set $S \subset \text{dom}V \subset \mathbf{R}^n$, the pair (V, S) is called a good solution of the dynamic programming inequality (3.8) if it satisfies

(i) V is a solution of the dynamic programming inequality (3.8) and there exists $\mathbf{u}^* : S \rightarrow \mathbf{U}$ such that

$$\begin{aligned} &\sup_{w \in \mathbf{W}} \max\{g(x, \mathbf{u}^*(x), w), V(f(x, \mathbf{u}^*(x), w))\} \\ &= \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} \max\{g(x, u, w), V(f(x, u, w))\}, \forall x \in S. \end{aligned} \quad (3.19)$$

(ii) S is an invariant set under the closed-loop dynamics when the controller is $\mathbf{u}^*(x)$, i.e. $\forall x \in S, \forall w \in \mathbf{W}, f(x, \mathbf{u}^*(x), w) \in S$.

A controller $K^* \in \mathcal{K}_{state}$ can be defined by $K^*(x)_k = \mathbf{u}^*(x_k)$, for $x \in \mathcal{X}_{0,\infty}$ (static state feedback). In this definition, if $S \neq \mathbf{R}^n$ we specify $\mathbf{u}^*(x)$ arbitrarily for $x \notin S$.

Theorem 3.5 *If (V, S) is a good solution of the dynamic programming inequality (3.8), then the closed-loop system (3.4) (with $K = K^*$) satisfies*

$$z_k \leq V(x_0), \quad \forall x_0 \in S, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0.$$

Moreover, if $B_0 \subset S$, then the closed-loop system is LIB dissipative with respect to B_0 .

PROOF. From (3.8) and (3.19),

$$V(x) \geq \sup_{w \in \mathbf{W}} V(f(x, \mathbf{u}^*(x), w)), \quad \forall x \in S.$$

Hence when $\xi_0 = x_0 \in S$, for the closed-loop system with $K = K^*$,

$$V(\xi_k) \geq V(\xi_{k+1}), \quad \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$$

and certainly

$$V(\xi_k) \leq V(x_0).$$

Also from (3.8) and (3.19), we have

$$g(\xi_k, \mathbf{u}^*(\xi_k), w_k) \leq V(\xi_k), \quad \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k},$$

hence

$$z_k = g(\xi_k, \mathbf{u}^*(\xi_k), w_k) \leq V(x_0), \quad \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}.$$

Moreover, if $B_0 \subset S$, then from Definition 2.2, the closed-loop system is LIB dissipative with respect to B_0 where $\beta(x_0) = V(x_0)$. \square

Corollary 3.6 *If (V, S) is a good solution of the dynamic programming inequality (3.8), then we have*

$$S \subset \text{dom}V_a; \quad V_a(x) \leq V(x), \quad \forall x \in S. \quad (3.20)$$

where V_a is the state feedback value function defined in (3.6).

PROOF. From the proof of Theorem 3.5, $\forall x_0 \in S$,

$$z_k = g(\xi_k, \mathbf{u}^*(\xi_k), w_k) \leq V(x_0), \quad \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}.$$

Hence

$$V_a(x_0) \leq \beta_a^{K^*}(x_0) \leq V(x_0) < +\infty,$$

where β_a^K is defined in (3.5). Hence (3.20) holds. \square

Remark 3.7 We know from Theorem 3.3 that $(V = V_a, S = \text{dom}V_a)$ is a solution to the dynamic programming inequality (3.8), so it follows that if $(V_a, \text{dom}V_a)$ is in fact a good solution (the “goodness” depends on the attainment of the infimum in item (i) of Definition 3.4), then the controller $K_a^*(x)_k = \mathbf{u}_a^*(x_k)$ obtained from it achieves the best LIB performance possible (i.e. it achieves the smallest bound $\beta = V_a$ possible in (2.4) (see Definition 2.2)). \square

4 Measurement Feedback Synthesis

We turn now to the measurement feedback synthesis problem for the LIB dissipation property. The solution to the problem requires suitable state estimation, in addition to some *concepts* from the state feedback solution.

4.1 Problem Statement

Consider nonlinear discrete-time system:

$$\begin{aligned}\xi_{k+1} &= f(\xi_k, u_k, w_k) \\ z_k &= g(\xi_k, u_k, w_k) \\ y_k &= h(\xi_k, u_k, w_k)\end{aligned}\tag{4.1}$$

Here, $\xi_k \in \mathbf{R}^n$, $u_k \in \mathbf{U} \subset \mathbf{R}^m$, $w_k \in \mathbf{W} \subset \mathbf{R}^s$, $y_k \in \mathbf{R}^p$ and $z_k \in \mathbf{R}$ are the state, control input, disturbance input, output and performance measure, respectively.

We continue to make Assumption 3.1. Besides (2.3) and (3.3), we employ the following notation:

$$\begin{aligned}\mathbf{Y} &= \text{range}\{h\}, \\ y_{0,k-1} &= \{y_0, \dots, y_{k-1}\}, \forall k \geq 0, \\ \mathcal{Y}_{0,k-1} &= \{y_{0,k-1} : y_i \in \mathbf{Y}, 0 \leq i \leq k-1\}, \\ \mathcal{Y}_{0,\infty} &= \{y_{0,\infty} : y_i \in \mathbf{Y}\}\end{aligned}\tag{4.2}$$

An *admissible* measurement feedback controller is a causal map $K : \mathcal{Y}_{0,\infty} \rightarrow \mathcal{U}_{0,\infty}$, meaning that for each time $k > 0$ if $y^1, y^2 \in \mathcal{Y}_{0,\infty}$ and $y_l^1 = y_l^2$ for all $0 \leq l \leq k-1$ then $K(y^1)_k = K(y^2)_k$, i.e., the control at time k is independent of current and future measurements. Denote by \mathcal{K} the class of such admissible controllers. We sometimes abuse notation by writing $u_k = K(y_{0,k-1})$ or $u = K(y)$.

Problem: Given $B_0 \subset \mathbf{R}^n$, find a measurement feedback controller $K \in \mathcal{K}$ such that the closed-loop system is LIB dissipative with respect to B_0 .

For a given controller $K \in \mathcal{K}$, the available storage of the closed-loop system is

$$\beta_a^K(x_0) = \sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} g(\xi_k, u_k, w_k), \forall x_0 \in \mathbf{R}^n\tag{4.3}$$

where ξ denotes the state trajectory of (4.1) with input $u = K(y)$, disturbance w and initial condition $\xi_0 = x_0$.

Problem Restatement: Given $B_0 \subset \mathbf{R}^n$, choose $K \in \mathcal{K}$ such that $B_0 \subset \text{dom}\beta_a^K$.

4.2 Problem Restatement in Terms of a Cost Function

We will solve the optimal K problem using a minimax cost function and information state methods in subsequent sections. The aim of this section is to define a suitable cost function and relate it to LIB dissipation.

Denote

$$\tilde{\mathbf{R}} \triangleq \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}. \quad (4.4)$$

and

$$\tilde{\chi} \triangleq \{p : \mathbf{R}^n \rightarrow \tilde{\mathbf{R}}\}. \quad (4.5)$$

For $p \in \tilde{\chi}$, denote

$$\langle p \rangle \triangleq \sup_{x \in \mathbf{R}^n} p(x) \quad (4.6)$$

Notice that $-\infty \leq \langle p \rangle \leq +\infty$. For $p \in \tilde{\chi}$, denote

$$\text{support } p \triangleq \{x \in \mathbf{R}^n : p(x) > -\infty\} \quad (4.7)$$

Also, for nonempty set $M \subset \mathbf{R}^n$, denote

$$\delta_M(x) \triangleq \begin{cases} 0, & \text{if } x \in M, \\ -\infty, & \text{if } x \notin M. \end{cases} \quad (4.8)$$

Then $\delta_M \in \tilde{\chi}$, $\langle \delta_M \rangle = 0$ and $\text{support } \delta_M = M$.

Define, for $p \in \tilde{\chi}$ and $u \in \mathbf{U}$, $y \in \mathbf{Y}$,

$$G(p, u, y) \triangleq \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p(x) + g(x, u, w) : h(x, u, w) = y\}. \quad (4.9)$$

Similar as $\check{g}(x)$ in the state feedback case, for $p \in \tilde{\chi}$, we define

$$\check{G}(p) \triangleq \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} G(p, u, y). \quad (4.10)$$

Remark 4.1 Notice that for $p \in \tilde{\chi}$,

$$\begin{aligned} \check{G}(p) &= \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} G(p, u, y) \\ &= \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p(x) + g(x, u, w) : h(x, u, w) = y\} \\ &= \inf_{u \in \mathbf{U}} \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p(x) + g(x, u, w)\} \\ &= \inf_{u \in \mathbf{U}} \sup_{x \in \mathbf{R}^n} \{p(x) + \sup_{w \in \mathbf{W}} g(x, u, w)\} \\ &\geq \sup_{x \in \mathbf{R}^n} \{p(x) + \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} g(x, u, w)\} \\ &\geq \sup_{x \in \mathbf{R}^n} \{p(x) + \check{g}(x)\} \\ &= \langle p + \check{g} \rangle. \end{aligned} \quad (4.11)$$

Hence from Assumption 3.1,

$$\check{G}(p) = -\infty \Leftrightarrow p \equiv -\infty.$$

□

For $p \in \tilde{\chi}$, controller K , define the *cost function*

$$J_p(K) \triangleq \sup_{k \geq 0} \sup_{x_0 \in \mathbf{R}^n} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\} \quad (4.12)$$

where ξ denotes the state trajectory of (4.1) with input $u = K(y)$, disturbance w and initial condition $\xi_0 = x_0$.

Lemma 4.2 *For $p \in \tilde{\chi}$, the cost function satisfies*

$$J_p(K) = \sup_{x_0 \in \text{support } p} \{p(x_0) + \beta_a^K(x_0)\}. \quad (4.13)$$

PROOF. By the definition of $J_p(K)$,

$$J_p(K) = \sup_{x_0 \in \mathbf{R}^n} \sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\}.$$

If $x_0 \notin \text{support } p$, then $p(x_0) = -\infty$ and

$$\sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\} = -\infty.$$

If $x_0 \in \text{support } p$, then

$$\sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \{p(x_0) + g(\xi_k, u_k, w_k)\} = p(x_0) + \beta_a^K(x_0).$$

Hence the equality (4.13) holds. □

Similar with [10], the cost function $J_p(K)$ have the following property.

Lemma 4.3 *The function $p \rightarrow J_p(K)$ satisfies*

(i) *Domination.* $J_p(K) \geq \check{G}(p)$, where $\check{G}(p)$ is defined in (4.10); $J_p(K) = -\infty \Leftrightarrow p \equiv -\infty$.

(ii) *Monotonicity.* $p_1 \geq p_2 \implies J_{p_1}(K) \geq J_{p_2}(K)$.

(iii) *Additive homogeneity.* $\forall c \in \mathbf{R}, J_{p+c}(K) = J_p(K) + c$.

PROOF. (i) By Lemma 4.2,

$$\begin{aligned} J_p(K) &= \sup_{x_0 \in \text{support } p} \{p(x_0) + \beta_a^K(x_0)\} \\ &\geq \sup_{x_0 \in \text{support } p} \left\{ p(x_0) + \sup_{w \in \mathbf{W}} g(x_0, u_0^K, w) \right\} \\ &\geq \inf_{u \in \mathbf{U}} \sup_{x_0 \in \text{support } p} \left\{ p(x_0) + \sup_{w \in \mathbf{W}} g(x_0, u, w) \right\} \\ &= \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} G(p, u, y) \\ &= \check{G}(p). \end{aligned}$$

Also by Lemma 4.2, $J_p(K) = -\infty \Leftrightarrow p \equiv -\infty$.

(ii) and (iii) are obvious from Lemma 4.2. □

Lemma 4.4 *The closed-loop system (with controller $K \in \mathcal{K}$) is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$, if and only if $\exists p \in \tilde{\chi}$ with support $p = B_0$ such that*

$$J_p(K) \leq 0. \quad (4.14)$$

PROOF. From Lemma 4.2,

$$\begin{aligned} J_p(K) \leq 0 &\Leftrightarrow \sup_{x_0 \in \text{support } p} \{p(x_0) + \beta_a^K(x_0)\} \leq 0 \\ &\Leftrightarrow \beta_a^K(x_0) \leq -p(x_0), \forall x_0 \in \text{support } p \\ &\Leftrightarrow \beta_a^K(x_0) \leq -p(x_0) < +\infty, \forall x_0 \in B_0 \end{aligned}$$

□

Problem Restatement I: Given $B_0 \subset \mathbf{R}^n$, choose controller $K \in \mathcal{K}$ such that

$$J_p(K) \leq 0 \quad (4.15)$$

for some $p \in \tilde{\chi}$ with support $p = B_0$.

Optimal K Problem I: Choose controller $K \in \mathcal{K}$ such that $J_p(K)$ is the smallest (over \mathcal{K}) for some $p \in \tilde{\chi}$ with support $p = B_0$.

4.3 Equivalent Formulation Using Information States

To solve the LIB problem, we introduce a state estimator quantity from which a suitable controller can be determined. This state quantity must be computable from the measurements (u, y) available to the controller, and it must characterize the LIB property. The information state framework of [14], [15], [10] is employed, and we recast the measurement feedback LIB dissipation problem in terms of an equivalent state feedback problem, where the *new* state is an information state.

4.3.1 Information States and LIB Dissipation

Definition 4.5 *For $p_0 \in \tilde{\chi}$, $j \geq 1$, $u_{0,j-1} \in \mathcal{U}_{0,j-1}$, $y_{0,j-1} \in \mathcal{Y}_{0,j-1}$, we define the information state $p_j : \mathbf{R}^n \rightarrow \tilde{\mathbf{R}}$ by*

$$p_j(x) = \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) : \xi_0 = x_0, \xi_j = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq j-1\} \quad (4.16)$$

where ξ_i satisfies

$$\xi_{i+1} = f(\xi_i, u_i, w_i), \quad 0 \leq i \leq j-1. \quad (4.17)$$

Remark 4.6 Notice that information state p_j is a function, $p_j(x)$ is the maximal value of all the possible $p_0(x_0)$. Since $p_j(x) \neq -\infty$ if and only if $\exists x_0, w_{0,j-1}$ such that

$$p_0(x_0) \neq -\infty, \xi_0 = x_0, \xi_j = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq j-1.$$

support p_j is the set of all the possible states at step j , and $p_j(x) = -\infty$ means that it is impossible to arrive x at step j . □

The next lemma shows that the information state characterizes the LIB dissipation property.

Lemma 4.7 *The closed-loop system (with controller $K \in \mathcal{K}$) is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$, if and only if $\exists p_0 \in \tilde{\chi}$ with support $p_0 = B_0$ such that*

$$G(p_k, u_k, y_k) \leq 0, \quad \forall k \geq 0, \quad (4.18)$$

where p_k is defined by (4.16) with initial state p_0 and $u = K(y)$, y is the output of (4.1) for any initialization $x_0 \in B_0$ and disturbance w , and $G(p, u, y)$ is defined in (4.9).

PROOF. Assume (4.18) holds as stated. $\forall x_0 \in B_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$, denote by ξ_k, u_k and y_k the corresponding state, control and output trajectories of the closed-loop system. By Definition 4.5 we have

$$p_0(x_0) + g(\xi_k, u_k, w_k) \leq p_k(\xi_k) + g(\xi_k, u_k, w_k) \leq G(p_k, u_k, y_k) \leq 0.$$

i.e.

$$z_k = g(\xi_k, u_k, w_k) \leq -p_0(x_0).$$

This implies that the closed-loop system is LIB dissipative with respect to B_0 where $\beta(x_0) = -p_0(x_0)$.

Conversely, suppose that the closed-loop system is LIB dissipative with respect to B_0 and $\beta(x_0)$. Choose $p_0(x_0) = -\beta(x_0)$, then support $p_0 = B_0$ and $\forall k \geq 0$,

$$\begin{aligned} & G(p_k, u_k, y_k) \\ &= \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p_k(x) + g(x, u_k, w) : h(x, u_k, w) = y_k\} \\ &= \sup_{x \in \mathbf{R}^n} \sup_{w_k \in \mathbf{W}} \{ \sup_{w_{0,k-1}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) : \xi_0 = x_0, \xi_k = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k-1\} \\ &\quad + g(x, u_k, w_k) : h(x, u_k, w_k) = y_k \} \\ &= \sup_{w_{0,k}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + g(\xi_k, u_k, w_k) : \xi_0 = x_0, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k\} \\ &= \sup_{w_{0,k}} \sup_{x_0 \in B_0} \{g(\xi_k, u_k, w_k) - \beta(x_0) : \xi_0 = x_0, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k\} \\ &\leq 0. \end{aligned}$$

□

4.3.2 Recursion

In this subsection we show that the information state satisfies a recursion of the form

$$p_{j+1}(x) = F(p_j, u_j, y_j)(x), \quad j \geq 0 \quad (4.19)$$

In fact, $\forall j \geq 0$,

$$\begin{aligned}
& p_{j+1}(x) \\
&= \sup_{w_{0,j} \in \mathcal{W}_{0,j}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) : \xi_0 = x_0, \xi_{j+1} = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq j\} \\
&= \sup_{w_j \in \mathbf{W}} \sup_{\xi_j \in \mathbf{R}^n} \left\{ \sup_{w_{0,j-1} \in \mathcal{W}_{0,j-1}} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) : \xi_0 = x_0, \xi_j = \xi_j, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq j-1\} \right. \\
&\quad \left. : \xi_{j+1} = f(\xi_j, u_j, w_j) = x, h(\xi_j, u_j, w_j) = y_j \right\} \\
&= \sup_{w_j \in \mathbf{W}} \sup_{\xi_j \in \mathbf{R}^n} \{p_j(\xi_j) : f(\xi_j, u_j, w_j) = x, h(\xi_j, u_j, w_j) = y_j\} \\
&= \sup_{w \in \mathbf{W}} \sup_{\xi \in \mathbf{R}^n} \{p_j(\xi) : f(\xi, u_j, w) = x, h(\xi, u_j, w) = y_j\}
\end{aligned}$$

Denote

$$F(p, u, y)(x) = \sup_{w \in \mathbf{W}} \sup_{\xi \in \mathbf{R}^n} \{p(\xi) : f(\xi, u, w) = x, h(\xi, u, w) = y\} \quad (4.20)$$

then for any given $p_0 \in \tilde{\chi}$, the information states can be obtained by the recursion (4.19).

Remark 4.8 The analogous partial differential equation for the continuous time system $\dot{\xi} = f(\xi, u, w)$, $z = g(\xi, u, w)$, $y = h(\xi, u, w)$ is

$$\frac{\partial}{\partial t} p_t(x) = \sup \{-\nabla p_t(x) \cdot f(x, u, w) : w \in \mathbf{W} \text{ and } y = h(x, u, w)\} \quad (4.21)$$

Thus

$$\dot{p} = F(p, u, y)$$

where F is defined by the RHS of (4.21). □

4.3.3 Equivalent Problem

In the definition of information state, u and y are independent. Now consider the case when the controller K is known, and $u = K(y)$.

For $p \in \tilde{\chi}$, define

$$\bar{J}_p(K) \triangleq \sup_{k \geq 0} \sup_{y_{0,k} \in \mathcal{Y}_{0,k}} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\} \quad (4.22)$$

where $p_k(x)$ are information states obtained by (4.16), and $G(p, u, y)$ is defined in (4.9).

For a given controller $K \in \mathcal{K}$ and $B \subset \mathbf{R}^n$, denote

$$\mathcal{Y}_{0,k}(K, B) \triangleq \{y_{0,k} : \exists x_0 \in B, \exists w_{0,k} \in \mathcal{W}_{0,k}, \text{ s.t. } y_i = h(\xi_i, u_i, w_i), 0 \leq i \leq k\}. \quad (4.23)$$

i.e. $\mathcal{Y}_{0,k}(K, B)$ denotes all the possible output $y_{0,k}$ of the closed-loop system (with controller $K \in \mathcal{K}$) where the initial state x_0 contains in the set B .

Remark 4.9 In the definition of $\bar{J}_p(K)$, the range of $y_{0,k}$ is not so important, in fact

$$\begin{aligned}\bar{J}_p(K) &= \sup_{k \geq 0} \sup_{y_{0,k}: y_i \in \mathbf{R}^p, 0 \leq i \leq k} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\} \\ &= \sup_{k \geq 0} \sup_{y_{0,k} \in \mathcal{Y}_{0,k}} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\} \\ &= \sup_{k \geq 0} \sup_{y_{0,k} \in \mathcal{Y}_{0,k}(K, \text{support } p)} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\}\end{aligned}$$

□

The relation between the cost function and the information state is given in the following theorem.

Theorem 4.10 *We have, for all $K \in \mathcal{K}$,*

$$\bar{J}_p(K) = J_p(K). \quad (4.24)$$

PROOF. Let $K \in \mathcal{K}$. Then

$$\begin{aligned}\bar{J}_p(K) &= \sup_{k \geq 0} \sup_{y_{0,k}} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\} \\ &= \sup_{k \geq 0} \sup_{y_{0,k}} \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p_k(x) + g(x, u_k, w) : h(x, u_k, w) = y_k\} \\ &= \sup_{k \geq 0} \sup_{y_{0,k-1}} \sup_{x \in \mathbf{R}^n} \sup_{w \in \mathbf{W}} \{p_k(x) + g(x, u_k, w)\} \\ &= \sup_{k \geq 0} \sup_{y_{0,k-1}} \sup_{x \in \mathbf{R}^n} \sup_{w_k \in \mathbf{W}} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \sup_{x_0 \in \mathbf{R}^n} \{p(x_0) + g(x, u_k, w_k) \\ &\quad : \xi_0 = x_0, \xi_k = x, h(\xi_i, u_i, w_i) = y_i, 0 \leq i \leq k-1\} \\ &= \sup_{k \geq 0} \sup_{w_k \in \mathbf{W}} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} \sup_{x_0 \in \mathbf{R}^n} \{p(x_0) + g(\xi_k, u_k, w_k) : \xi_0 = x_0\} \\ &= \sup_{k \geq 0} \sup_{w_{0,k} \in \mathcal{W}_{0,k}} \sup_{x_0 \in \mathbf{R}^n} \{p(x_0) + g(\xi_k, u_k, w_k) : \xi_0 = x_0\} \\ &= J_p(K)\end{aligned}$$

as required. □

Problem Restatement II: Given $B_0 \subset \mathbf{R}^n$, choose a controller $K \in \mathcal{K}$ such that

$$\bar{J}_p(K) \leq 0.$$

for some $p \in \tilde{\chi}$ with support $p = B_0$.

Optimal K Problem II: Choose a controller $K \in \mathcal{K}$ such that $\bar{J}_p(K)$ is the smallest for some $p \in \tilde{\chi}$ with support $p = B_0$.

4.4 Dynamic Programming Solution

In this section we show how to synthesize LIB dissipative controllers by finding optimal minimax controllers solving the Optimal K Problem II. We will make use of a dynamic programming equation and inequality analogous to (3.7) and (3.8).

Let $W : \tilde{\chi} \rightarrow \tilde{\mathbf{R}}$, and define

$$\text{dom}W \triangleq \{p \in \tilde{\chi} : -\infty < W(p) < +\infty\} \quad (4.25)$$

where $\tilde{\chi}$ is the function space defined by (4.5). Let $\tilde{S} \subset \text{dom}W \subset \tilde{\chi}$. The *measurement feedback dynamic programming equation* is

$$W(p) = \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{G(p, u, y), W(F(p, u, y))\}, \quad \forall p \in \tilde{S}, \quad (4.26)$$

where $G(p, u, y)$ and $F(p, u, y)$ are defined in (4.9) and (4.20). The *measurement feedback dynamic programming inequality* is

$$W(p) \geq \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{G(p, u, y), W(F(p, u, y))\}, \quad \forall p \in \tilde{S}. \quad (4.27)$$

Remark 4.11 Notice that (3.7), (3.8), and (4.26), (4.27) have the same *form*, respectively. \square

Remark 4.12 The analogous partial differential equation for the continuous time system $\dot{\xi} = f(\xi, u, w)$, $z = g(\xi, u, w)$, $y = h(\xi, u, w)$ is

$$\inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{G(p, u, y) - W(p), \nabla W(p)F(p, u, y)\} = 0 \text{ in } \tilde{S} \subset \text{dom}W. \quad (4.28)$$

\square

4.4.1 Necessity

For $p \in \tilde{\chi}$, define the *measurement feedback value function*

$$W_a(p) \triangleq \inf_{K \in \mathcal{K}} \bar{J}_p(K) = \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_{0,k} \in \mathcal{Y}_{0,k}} \{G(p_k, u_k, y_k) : p_0 = p, u = K(y)\} \quad (4.29)$$

where the minimization ranges over the class of all the admissible measurement feedback controllers \mathcal{K} . Notice that $W_a : \tilde{\chi} \rightarrow \tilde{\mathbf{R}}$, i.e., $-\infty \leq W_a(p) \leq +\infty$.

Theorem 4.13 *Assume that there exists an admissible measurement feedback controller K_0 such that the closed-loop system is LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$, $\beta_{K_0}(x_0)$. Then:*

(i) *The set $\text{dom}W_a$ is nonempty, $p_0 = -\beta_{K_0} \in \text{dom}W_a$.*

(ii) *The following structural properties hold:*

(ii-a) *W_a dominates \check{G} : $W_a(p) \geq \check{G}(p), \forall p \in \tilde{\chi}$, where $\check{G}(p)$ is defined in (4.10); $W_a(p) = -\infty \Leftrightarrow p(x) \equiv -\infty$.*

(ii-b) *W_a is monotone: $\forall p_1, p_2 \in \tilde{\chi}$, if $p_1 \geq p_2$, then $W_a(p_1) \geq W_a(p_2)$. Moreover, if $p_1 \in \text{dom}W_a$ and $\langle p_2 \rangle > -\infty$, then $p_2 \in \text{dom}W_a$.*

(ii-c) W_a is additive homogeneous: $\forall c \in \mathbf{R}, \forall p \in \text{dom}W_a, W_a(p + c) = W_a(p) + c$.

(iii) Fix $p \in \tilde{\chi}$ and assume $\bar{J}_p(K_0)$ is finite, then

$$\begin{aligned} \bar{J}_p(K_0) &\geq \max\left\{\max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j)\right\} \text{ and } p_j \in \text{dom}W_a, \\ \forall j &\geq 0, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_0, \text{support } p), u = K_0(y), p_0 = p. \end{aligned} \quad (4.30)$$

where $\mathcal{Y}_{0,k}(K, B)$ is defined by (4.23).

(iv) Fix $p \in \text{dom}W_a$, if K_ε is an ε -optimal controller (i.e., $\bar{J}_p(K_\varepsilon) \leq W_a(p) + \varepsilon$), then

$$\begin{aligned} W_a(p) + \varepsilon &\geq \max\left\{\max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j)\right\} \text{ and } p_j \in \text{dom}W_a, \\ \forall j &\geq 0, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_\varepsilon, \text{support } p), u = K_\varepsilon(y), p_0 = p. \end{aligned} \quad (4.31)$$

(v) The dynamic programming relation holds:

$$\begin{aligned} W_a(p) &= \inf_{K \in \mathcal{K}} \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max\left\{\max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j)\right\} : p_0 = p, u = K(y) \right\}, \\ &\quad \forall p \in \text{dom}W_a, \forall j \geq 0 \end{aligned} \quad (4.32)$$

i.e. W_a solves the dynamic programming equation (4.26), with $\tilde{S} = \text{dom}W_a$.

PROOF. (i) From Lemma 4.3 and Theorem 4.10, $\forall p \in \tilde{\chi}, \forall K, \bar{J}_p(K) = J_p(K) \geq \check{G}(p)$, hence

$$W_a(p) = \inf_{K \in \mathcal{K}} \bar{J}_p(K) \geq \check{G}(p).$$

For $p_0 = -\beta_{K_0}$,

$$-\infty < \check{G}(-\beta_{K_0}) = \check{G}(p_0) \leq W_a(p_0) = \inf_{K \in \mathcal{K}} \bar{J}_{p_0}(K) \leq \bar{J}_{p_0}(K_0) \leq 0 \quad (4.33)$$

hence $p_0 \in \text{dom}W_a$, so $\text{dom}W_a$ is nonempty.

(ii-a) The domination is proved in (i). If $p(x) \equiv -\infty$, then $\bar{J}_p(K) = -\infty$ for any K , hence $W_a(p) = -\infty$. If $\langle p \rangle > -\infty$, then $W_a(p) \geq \check{G}(p) > -\infty$.

For (ii-b), from Lemma 4.3, $p_1 \geq p_2 \Rightarrow \forall K, \bar{J}_{p_1}(K) \geq \bar{J}_{p_2}(K) \Rightarrow W_a(p_1) \geq W_a(p_2)$. Moreover, if $p_1 \in \text{dom}W_a$ and $\langle p_2 \rangle > -\infty$, then $-\infty < \check{G}(p_2) \leq W_a(p_2) \leq W_a(p_1) < \infty$, hence $p_2 \in \text{dom}W_a$. (Notice that for $p \in \tilde{\chi}, \langle p \rangle > -\infty$ if and only if $\check{G}(p) > -\infty$.)

(ii-c) is obvious since $\forall K \in \mathcal{K}, \bar{J}_{p+c}(K) = \bar{J}_p(K) + c$. (Lemma 4.3)

(iii) If $j = 0$, $\bar{J}_p(K_0) \geq W_a(p_0) \geq \check{G}(p_0) = \check{G}(p) > -\infty$, hence $p_0 \in \text{dom}W_a$. (If $p \equiv -\infty$, then $\bar{J}_p(K_0) = -\infty$ is not finite, so $\langle p \rangle > -\infty$)

Now fix $j > 0$ and fix $y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_0, \text{support } p)$, then we can obtain p_j by $p_0 = p$ and $u_i = K_0(y_{0,i-1}), 0 \leq i \leq j-1$. From $y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_0, \text{support } p)$, $\langle p_j \rangle > -\infty$.

For any $k \geq 0$ and any $\tilde{y}_{0,k}$, denote

$$\hat{y}_i = \begin{cases} y_i, & 0 \leq i \leq j-1 \\ \tilde{y}_{i-j}, & i \geq j \end{cases} \quad (4.34)$$

and

$$K_0^j(\tilde{y}_{0,k-1}) = K_0(\hat{y}_{0,k+j-1}), \quad k \geq 0. \quad (4.35)$$

Then we can prove that

$$\begin{aligned} \bar{J}_p(K_0) &= \sup_{k \geq 0} \sup_{\bar{y}_{0,k}} \{G(p_k, u_k, \bar{y}_k) : p_0 = p, u = K_0(\bar{y})\} \\ &\geq \sup_{k \geq 0} \sup_{\hat{y}_{0,k}} \{G(p_k, u_k, \hat{y}_k) : p_0 = p, u = K_0(\hat{y})\} \\ &\geq \max \left\{ \sup_{0 \leq k \leq j-1} \{G(p_k, u_k, y_k) : p_0 = p, u_i = K_0(y_{0,i-1}), 0 \leq i \leq j-1\}, \right. \\ &\quad \left. \sup_{k \geq j} \sup_{\hat{y}_{0,k}} \{G(p_k, u_k, \hat{y}_k) : p_0 = p, u = K_0(\hat{y})\} \right\} \\ &\geq \max \left\{ \left\{ \max_{0 \leq k \leq j-1} G(p_k, u_k, y_k) : p_0 = p, u_i = K_0(y_{0,i-1}), 0 \leq i \leq j-1 \right\}, \bar{J}_{p_j}(K_0^j) \right\} \\ &= \max \left\{ \left\{ \max_{0 \leq k \leq j-1} G(p_k, u_k, y_k) : p_0 = p, u_i = K_0(y_{0,i-1}), 0 \leq i \leq j-1 \right\}, \bar{J}_{p_j}(K_0^j) \right\} \\ &\geq \max \left\{ \left\{ \max_{0 \leq k \leq j-1} G(p_k, u_k, y_k) : p_0 = p, u_i = K_0(y_{0,i-1}), 0 \leq i \leq j-1 \right\}, W_a(p_j) \right\}. \end{aligned}$$

Hence

$$+\infty > \bar{J}_p(K_0) \geq W_a(p_j) \geq \check{G}(p_j) > -\infty$$

and $p_j \in \text{dom}W_a$.

(iv) Apply part (iii) with $K_0 = K_\varepsilon$.

(v) $\forall p \in \text{dom}W_a$, $W_a(p)$ is finite, $\forall \varepsilon > 0$, $\exists K_\varepsilon$ such that $\bar{J}_p(K_\varepsilon) \leq W_a(p) + \varepsilon$ (K_ε is an ε -optimal controller), from (iv), $\forall j \geq 0$,

$$\begin{aligned} W_a(p) + \varepsilon &\geq \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\}, \\ \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_\varepsilon, \text{support } p), p_0 = p, u = K_\varepsilon(y). \end{aligned}$$

hence

$$\begin{aligned} &W_a(p) + \varepsilon \\ &\geq \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_\varepsilon, \text{support } p)} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K_\varepsilon(y) \right\} \\ &= \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K_\varepsilon(y) \right\} \end{aligned}$$

certainly

$$W_a(p) + \varepsilon \geq \inf_{K \in \mathcal{K}} \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K(y) \right\}$$

since ε is arbitrary, we have

$$W_a(p) \geq \inf_{K \in \mathcal{K}} \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K(y) \right\}. \quad (4.36)$$

To prove the opposite inequality, for $p \in \text{dom}W_a$, $j \geq 0$, define

$$R(p) = \inf_{K \in \mathcal{K}} \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K(y) \right\}. \quad (4.37)$$

From (4.36), $R(p) \leq W_a(p) < +\infty$. Moreover, $R(p) \geq \check{G}(p) > -\infty$, So $R(p)$ is a finite number.

Let $\varepsilon > 0$, choose K^1 such that

$$\begin{aligned}
& R(p) \\
& \geq \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K^1(y) \right\} - \varepsilon \\
& = \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^1, \text{support } p)} \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K^1(y) \right\} - \varepsilon.
\end{aligned} \tag{4.38}$$

(Therefore we have $p_j \in \text{dom}W_a, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^1, \text{support } p), u = K^1(y)$.)

$\forall q \in \text{dom}W_a, \exists K_q^2$ such that

$$W_a(q) \geq \bar{J}_q(K_q^2) - \varepsilon.$$

Define K^3 by

$$K^3(y_{0,i-1}) = \begin{cases} K^1(y_{0,i-1}), & 0 \leq i \leq j-1 \\ K_{p_j}^2(y_{j,i-1}), & i \geq j \end{cases} \tag{4.39}$$

Then $\forall k \geq j, \forall y_{0,k} \in \mathcal{Y}_{0,k}(K^3, \text{support } p)$, we have $y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K^1, \text{support } p)$ and

$$\begin{aligned}
R(p) & \geq \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), W_a(p_j) \right\} : p_0 = p, u = K^1(y) \right\} - \varepsilon \\
& \geq \left\{ \max \left\{ \max_{0 \leq i \leq j-1} G(p_i, u_i, y_i), \bar{J}_{p_j}(K_{p_j}^2) \right\} : p_0 = p, u = K^1(y) \right\} - 2\varepsilon \\
& \geq \left\{ \sup_{0 \leq i \leq k} G(p_i, u_i, y_i) : p_0 = p, u = K^3(y) \right\} - 2\varepsilon.
\end{aligned} \tag{4.40}$$

So we have

$$R(p) \geq \bar{J}_p(K^3) - 2\varepsilon$$

Hence

$$W_a(p) \leq \bar{J}_p(K^3) \leq R(p) + 2\varepsilon$$

since ε is arbitrary, we have

$$W_a(p) \leq R(p). \tag{4.41}$$

From (4.36) and (4.41), the proof is completed. \square

4.4.2 Information State Controllers

In state feedback synthesis of Section 3.2.2, the optimal controller was obtained by minimizing the RHS of the dynamic programming equation or inequality over u , to yield a static state feedback controller. We follow the same procedure in the measurement feedback case (next subsection), and obtain an optimal controller defined in terms of a static function of the information state. This gives a dynamic controller, a causal function of the measurements, of the type we now describe.

Let \mathbf{u} be a function

$$\mathbf{u} : \tilde{\chi} \rightarrow \mathbf{R}^m. \tag{4.42}$$

Define

$$u_k = \mathbf{u}(p_k). \quad (4.43)$$

Since $p_{k+1} = F(p_k, u_k, y_k) = F(p_k, \mathbf{u}(p_k), y_k)$, the information state p_{k+1} can be regarded as a function of $y_{0,k}$ and u_k is a function of $y_{0,k-1}$. This leads us to define the measurement feedback controller $K_{p_0}^{\mathbf{u}} \in \mathcal{K}$ by

$$K_{p_0}^{\mathbf{u}}(y.)_k = K_{p_0}^{\mathbf{u}}(y_{0,k-1}) = \mathbf{u}(p_k). \quad (4.44)$$

This *information state feedback controller* is a special kind of measurement feedback controller.

The state space realization of the closed-loop system (with controller $K_{p_0}^{\mathbf{u}} \in \mathcal{K}$) is

$$\begin{cases} \xi_{k+1} = f(\xi_k, u_k, w_k) \\ z_k = g(\xi_k, u_k, w_k) \\ y_k = h(\xi_k, u_k, w_k) \\ p_{k+1} = F(p_k, u_k, y_k) \\ u_k = \mathbf{u}(p_k) \end{cases} \quad (4.45)$$

with initialization $\xi_0 = x_0, p_0 \in \tilde{\chi}$.

4.4.3 Sufficiency

We will show how to obtain the optimal information state controller from dynamic programming inequality.

Definition 4.14 *Given a function $W : \tilde{\chi} \rightarrow \tilde{\mathbf{R}}$ and an nonempty set $\tilde{S} \subset \text{dom}W \subset \tilde{\chi}$, the pair (W, \tilde{S}) is called a good solution of the dynamic programming inequality (4.27) if*

(i) *W is a solution of (4.27) and there exists $\mathbf{u}^* : \tilde{S} \rightarrow \mathbf{U}$ such that $\forall p \in \tilde{S}$,*

$$\begin{aligned} & \sup_{y \in \mathbf{Y}} \max\{G(p, \mathbf{u}^*(p), y), W(F(p, \mathbf{u}^*(p), y))\} \\ & = \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{G(p, u, y), W(F(p, u, y))\}. \end{aligned} \quad (4.46)$$

(ii) *\tilde{S} is an invariant set under the closed-loop dynamics when the controller is $\mathbf{u}^*(p), \forall p \in \tilde{S}$. i.e. $\forall p \in \tilde{S}, \forall y \in \mathcal{Y}_{0,0}(K_p^{\mathbf{u}^*}, \text{support } p), F(p, \mathbf{u}^*(p), y) \in \tilde{S}$.*

Theorem 4.15 *Assume that (W, \tilde{S}) is a good solution of the dynamic programming inequality (4.27) and $p_0 \in \tilde{S}$. Then the closed-loop system with the controller defined by*

$$u_k = \mathbf{u}^*(p_k) \quad (4.47)$$

satisfies $\forall x_0 \in \text{support } p_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$,

$$z_k \leq W(p_0) - p_0(x_0). \quad (4.48)$$

Moreover, if $B_0 \subset \text{support } p_0$, then the closed-loop system is LIB dissipative with respect to B_0 where $\beta(x_0) = W(p_0) - p_0(x_0)$.

PROOF. Let $K_{p_0}^{\mathbf{u}^*}$ denote the information state controller obtained by (4.47). (Notice that $K_{p_0}^{\mathbf{u}^*}$ depends on the initial information state p_0 though $\mathbf{u}^*(\cdot)$ doesn't.) Now from (4.27) and (4.46), we have

$$W(p) \geq \sup_{y \in \mathbf{Y}} W(F(p, \mathbf{u}^*(p), y)), \forall p \in \tilde{S}.$$

Hence $\forall j \geq 0, \forall y_{0,j-1} \in \mathcal{Y}_{0,j-1}(K_{p_0}^{\mathbf{u}^*}, \text{support } p_0)$,

$$W(p_i) \geq W(p_{i+1}), \quad 0 \leq i \leq j-1, p_0 = p_0, u = K_{p_0}^{\mathbf{u}^*}(y)$$

and certainly

$$W(p_j) \leq W(p_0).$$

Also from (4.27) and (4.46), we have

$$G(p_j, \mathbf{u}^*(p_j), y_j) \leq W(p_j), \quad \forall j \geq 0$$

hence

$$J_{p_0}(K_{p_0}^{\mathbf{u}^*}) = \bar{J}_{p_0}(K_{p_0}^{\mathbf{u}^*}) = \sup_{j \geq 0} \sup_{y_{0,j}} G(p_j, \mathbf{u}^*(p_j), y_j) \leq W(p_0)$$

For the closed-loop system, $\forall x_0 \in \text{support } p_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$,

$$p_0(x_0) + z_k \leq p_k(\xi_k) + g(\xi_k, \mathbf{u}^*(p_k), w_k) \leq G(p_k, \mathbf{u}^*(p_k), y_k) \leq J_{p_0}(K_{p_0}^{\mathbf{u}^*}) \leq W(p_0)$$

and hence (4.48) holds. Moreover, when $B_0 \subset \text{support } p_0$,

$$z_k \leq W(p_0) - p_0(x_0), \quad \forall x_0 \in B_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}.$$

Therefore the closed-loop system is LIB dissipative with respect to B_0 where $\beta(x_0) = W(p_0) - p_0(x_0)$. \square

Corollary 4.16 *If (W, \tilde{S}) is a good solution of the dynamic programming inequality (4.27), then we have*

$$\tilde{S} \subset \text{dom } W_a, \quad W_a(p) \leq W(p), \forall p \in \tilde{S}. \quad (4.49)$$

where W_a is the measurement feedback value function defined in (4.29).

PROOF. From the proof of Theorem 4.15, $\forall p_0 \in \tilde{S}$,

$$\bar{J}_{p_0}(K_{p_0}^{\mathbf{u}^*}) = J_{p_0}(K_{p_0}^{\mathbf{u}^*}) \leq W(p_0).$$

Hence

$$W_a(p_0) \leq \bar{J}_{p_0}(K_{p_0}^{\mathbf{u}^*}) \leq W(p_0) < +\infty.$$

Since we also have $W_a(p_0) \geq \check{G}(p_0) > -\infty$, (4.49) holds. \square

Remark 4.17 We know from Theorem 4.13 that $(W_a, \text{dom}W_a)$ is a solution to the dynamic programming inequality (4.27), so it follows that if $(W_a, \text{dom}W_a)$ is in fact a *good solution*, then the controller $K_a^*(y)_k = \mathbf{u}_a^*(p_k)$ obtained from it achieves the best LIB performance possible (i.e. it achieves the smallest bound $\beta(x_0) = W_a(p_0) - p_0(x_0)$ possible in (2.4) (see Definition 2.2). If the LIB dissipation control problem is solvable by *some* measurement feedback controller, then it is also solvable by an information state feedback controller whenever a good solution to the dynamic programming inequality (4.27) exists. $(W_a, \text{dom}W_a)$ will be a solution, and its “goodness” depends on the attainment of the infimum in item (i) of Definition 4.14. \square

Remark 4.18 It will be of particular interest to find good solutions (W, \tilde{S}) with \tilde{S} finite dimensional (see Sections 5.1.2 and 5.2.2). \square

4.5 Special Case: Uniform LIB

In [22], the authors formulate the problem of obtaining *uniform* bounds on the LIB performance. In this section, we consider this uniform LIB case. The results of Section 4.4 simplify, and the connection with the Shamma-Tu separation structure [22] is given.

Problem: Given $B_0 \subset \mathbf{R}^n$, find an measurement feedback controller $K \in \mathcal{K}$ such that the closed-loop system is *uniform LIB dissipative with respect to* B_0 , i.e. there exists a $\beta \in \mathbf{R}$ such that

$$z_k \leq \beta, \quad \forall x_0 \in B_0, \forall w_{0,k} \in \mathcal{W}_{0,k}, \forall k \geq 0. \quad (4.50)$$

For this special problem, since β does not depend on x_0 , we can constrain the information states in the subset

$$\bar{S} \triangleq \{\delta_X : X \subset \mathbf{R}^n\} \subset \tilde{\chi}, \quad (4.51)$$

where δ_M is defined in (4.8). Notice that \bar{S} is an invariant set under the recursion (4.19). Write

$$\bar{S}' \triangleq \{\text{subsets of } \mathbf{R}^n\}. \quad (4.52)$$

Then it is easy to check that $F(\delta_X, u, y) \in \bar{S}$ for all $X \in \bar{S}'$, $u \in \mathbf{U}$ and $y \in \mathbf{Y}$.

Now we choose the initial information state

$$p_0(x) = \delta_{X_0}(x)$$

where $X_0 \subset \mathbf{R}^n$ (i.e. $X_0 \in \bar{S}'$). Then we have

$$p_k(x) = \delta_{X_k}(x)$$

where

$$X_k = \text{support } p_k \quad (4.53)$$

and $x \in X_k$ if and only if there exists a trajectory ξ . of (4.1) with $\xi_0 \in X_0$ that is consistent with the given signals u, y (satisfying (4.1)). This is the *set-valued observer* in [22, page 259].

The set-valued state estimate X_k can be computed from the recursion

$$X_{k+1} = \hat{F}(X_k, u_k, y_k) \triangleq F(\delta_{X_k}, u_k, y_k) \quad (4.54)$$

where, given $X \in \bar{S}'$, $u \in \mathbf{U}$, $y \in \mathbf{Y}$, $x \in \hat{F}(X, u, y)$ if and only if there exists $x' \in X$, $w \in \mathbf{W}$ such that $f(x', u, w) = x$ and $h(x', u, w) = y$ (cf. [22, equation (6)]).

These considerations allow us to restrict our attention to the “smaller” space $\bar{S}' \approx \bar{S}$, in place of $\tilde{\chi}$. Any function $W : \tilde{\chi} \rightarrow \tilde{\mathbf{R}}$ projects to (or defines) a function $\hat{W} : \bar{S}' \rightarrow \tilde{\mathbf{R}}$ via

$$\hat{W}(X) \triangleq W(\delta_X)$$

for $X \in \bar{S}'$. Now, denote

$$\hat{G}(X, u, y) \triangleq G(\delta_X, u, y), \quad (4.55)$$

and

$$\check{G}(X) \triangleq \check{G}(\delta_X). \quad (4.56)$$

Let $\hat{W} : \bar{S}' \rightarrow \tilde{\mathbf{R}}$, and define

$$\text{dom } \hat{W} \triangleq \left\{ X \in \bar{S}' : -\infty < \hat{W}(X) < +\infty \right\}. \quad (4.57)$$

Let $\hat{S} \subset \text{dom } \hat{W} \subset \bar{S}'$. The measurement feedback dynamic programming equation (4.26) becomes

$$\hat{W}(X) = \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{\hat{G}(X, u, y), \hat{W}(\hat{F}(X, u, y))\}, \quad \forall X \in \hat{S}. \quad (4.58)$$

The measurement feedback dynamic programming inequality (4.27) becomes

$$\hat{W}(X) \geq \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max\{\hat{G}(X, u, y), \hat{W}(\hat{F}(X, u, y))\}, \quad \forall X \in \hat{S}. \quad (4.59)$$

The results of Section 4.4 become the following.

Lemma 4.19 *The closed-loop system (with controller $K \in \mathcal{K}$) is uniform LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$ if and only if there exists $\beta \in \mathbf{R}$ such that*

$$\hat{G}(X_k, u_k, y_k) \leq \beta, \quad \forall k \geq 0, \quad (4.60)$$

where X_k is defined by (4.54) with initial state $X_0 = B_0$, $u = K(y)$ and y is the output of (4.1) for any initialization $x_0 \in B_0$ and disturbance w .

We define the value function $\hat{W}_a : \bar{S}' \rightarrow \tilde{\mathbf{R}}$ by

$$\hat{W}_a(X) \triangleq \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_0, k \in \mathcal{Y}_{0, k}} \left\{ \hat{G}(X_k, u_k, y_k) : X_0 = X, u = K(y) \right\}. \quad (4.61)$$

Theorem 4.20 *(Necessity) Assume that there exists a controller K_0 such that the closed-loop system is uniform LIB dissipative with respect to $B_0 \subset \mathbf{R}^n$. Then the value function $\hat{W}_a(X)$ defined by (4.61) satisfies:*

- (i) $\text{dom}\hat{W}_a$ is nonempty, $B_0 \in \text{dom}\hat{W}_a$;
- (ii) $\hat{W}_a(X) \geq \check{G}(X), \forall X \subset \mathbf{R}^n$, where $\check{G}(X)$ is defined in (4.56). $\hat{W}_a(X) = -\infty \Leftrightarrow X = \emptyset$; $X_1 \subset X_2 \Rightarrow \hat{W}_a(X_1) \leq \hat{W}_a(X_2)$, if $X_1 \subset X_2, X_1 \neq \emptyset$ and $X_2 \in \text{dom}\hat{W}_a$, then $X_1 \in \text{dom}\hat{W}_a$;
- (iii) $\hat{W}_a(X)$ satisfies the dynamic programming relation

$$\hat{W}_a(X) = \inf_{K \in \mathcal{K}} \sup_{y_{0,j-1} \in \mathcal{Y}_{0,j-1}} \{ \max_{0 \leq i \leq j-1} \{ \hat{G}(X_i, u_i, y_i), \hat{W}_a(X_j) \} : X_0 = X, u = K(y) \},$$

$$\forall X \in \text{dom}\hat{W}_a, \forall j \geq 0. \quad (4.62)$$

i.e. $\hat{W}_a(X)$ is a solution of the dynamic programming equation (4.58) with $\hat{S} = \text{dom}\hat{W}_a$.

Definition 4.21 Given a function $\hat{W} : \bar{S}' \rightarrow \bar{\mathbf{R}}$ and a nonempty set $\hat{S} \subset \text{dom}\hat{W} \subset \bar{S}'$, the pair (\hat{W}, \hat{S}) is said to be a good solution of the dynamic programming inequality (4.59) provided

- (i) \hat{W} is a solution of (4.59) and $\forall X \in \hat{S}$, there exists $\hat{u}^* : \hat{S} \rightarrow \mathbf{U}$ such that

$$\begin{aligned} & \sup_{y \in \mathbf{Y}} \max \{ \hat{G}(X, \hat{u}^*(X), y), \hat{W}(\hat{F}(X, \hat{u}^*(X), y)) \} \\ &= \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \max \{ \hat{G}(X, u, y), \hat{W}(\hat{F}(X, u, y)) \}, \forall X \in \hat{S}. \end{aligned} \quad (4.63)$$

- (ii) \hat{S} is an invariant set under the closed-loop dynamics when the controller is $\hat{u}^*(X), \forall X \in \hat{S}$. i.e. $\forall X \in \hat{S}, \forall y \in \mathcal{Y}_{0,0}(K_{X^*}^{\hat{u}^*}, X), \hat{F}(X, \hat{u}^*(X), y) \in \hat{S}$.

Theorem 4.22 (Sufficiency) Assume that (\hat{W}, \hat{S}) is a good solution of the dynamic programming inequality (4.59) and $X_0 \in \hat{S}$. Then the closed-loop system with the controller defined by

$$u_k = \hat{u}^*(X_k) \quad (4.64)$$

satisfies $\forall x_0 \in X_0, \forall k \geq 0, \forall w_{0,k} \in \mathcal{W}_{0,k}$,

$$z_k \leq \hat{W}(X_0). \quad (4.65)$$

Moreover, if $B_0 \subset X_0$, then the closed-loop system is uniform LIB dissipative with respect to B_0 and $\beta = \hat{W}(X_0)$.

Corollary 4.23 If (\hat{W}, \hat{S}) is a good solution of the dynamic programming inequality (4.59), then we have

$$\hat{S} \subset \text{dom}\hat{W}_a, \quad \hat{W}_a(X) \leq \hat{W}(X), \forall X \in \hat{S}. \quad (4.66)$$

where \hat{W}_a is the value function defined in (4.61).

Remark 4.24 The above results are consistent with [22, Theorem 4.1] which asserts, in our terminology, that if there exists a measurement feedback controller achieving the uniform LIB dissipation property, then there exists a separation structure controller that feeds back the set-valued observer state and also achieves uniform LIB dissipation. \square

5 Examples

5.1 Example 1 - A System with Linear Dynamics

Consider one-dimensional discrete-time system with linear dynamics:

$$\begin{cases} \xi_{k+1} = a\xi_k + bu_k + w_k \\ z_k = |\xi_k| \\ y_k = \xi_k + w_k \end{cases} \quad (5.1)$$

where $\xi_k, w_k, u_k, y_k, z_k \in \mathbf{R}, 1 < a \leq 2, b > 0$. Notice that when $u_k = w_k = 0, k \geq 0$, the open-loop system is unstable.

We consider the uniform LIB dissipation problem described in Section 4.5. Suppose $B_0 = [a_{0l}, a_{0r}] \subset \mathbf{R}, \mathbf{W} = [-d, d], d > 0, \mathbf{U} = [-\delta, \delta], \delta > 0$.

5.1.1 State Feedback Synthesis

Assume that

$$\delta \geq \frac{ad}{b}; \quad \max\{|a_{0l}|, |a_{0r}|\} \leq \frac{b\delta - d}{a - 1}. \quad (5.2)$$

The dynamic programming equation (3.7) was solved analytically for the value function $V_a(x)$ and corresponding state feedback controller $\mathbf{u}^*(x)$. Detailed calculations are provided in Appendix A.1.1. The value function is given by

$$V_a(x_0) = \max\{|x_0|, d\}, \quad \forall x_0 \in \text{dom}V_a = \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right], \quad (5.3)$$

and the corresponding optimal state feedback controller is

$$\mathbf{u}^*(x) \triangleq \begin{cases} \delta, & \text{if } x \in (-\infty, -\frac{b\delta}{a}) \\ -\frac{a}{b}x, & \text{if } x \in [-\frac{b\delta}{a}, \frac{b\delta}{a}] \\ -\delta, & \text{if } x \in (\frac{b\delta}{a}, +\infty) \end{cases} \quad (5.4)$$

The fact that this analytical solution of (3.7) is the value function V_a is proven in Appendix A.1.1 (such equations could have multiple solutions).

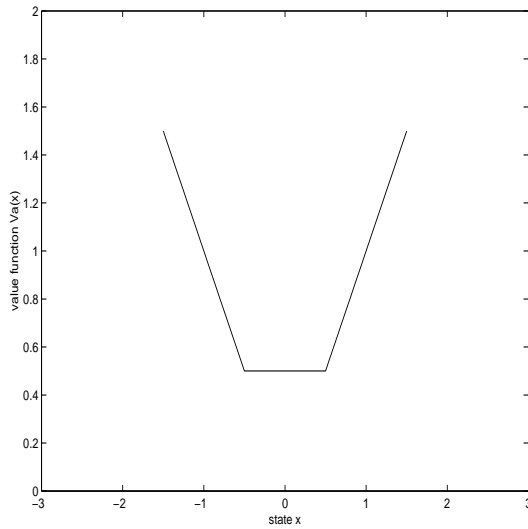
The dynamic programming equation (3.7) was also solved numerically ($a = 2, b = 1, d = 0.5, \delta = 2$). These numerical results correspond to the analytical solution above for the value function V_a and corresponding controller. The results are shown in (a), (c) Figure 5.1.

Theorem 3.5 asserts that the closed-loop system should be LIB dissipative. Indeed, the calculations in Appendix A.1.1 verify LIB dissipation with respect to $B_0 = [a_{0l}, a_{0r}]$ and $\beta = \max\{|a_{0l}|, |a_{0r}|, d\}$, provided that the assumption (5.2) holds. This is illustrated in (b), (d) Figure 5.1 ($B_0 = [-1.5, 1.5], \beta = 1.5, x_0 = 1.5, V_a(x_0) = 1.5$).

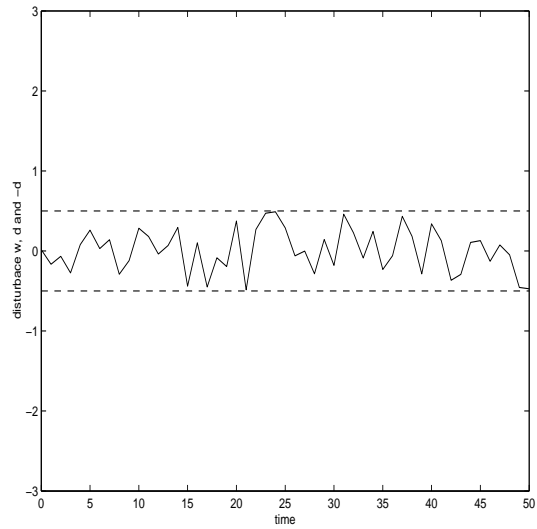
Remark 5.1 When $a = 1$, $\text{dom}V_a = \mathbf{R}$ and the optimal state feedback controller is

$$\mathbf{u}^*(x) = \begin{cases} \delta, & \text{if } x \in (-\infty, -b\delta) \\ -\frac{1}{b}x, & \text{if } x \in [-b\delta, b\delta] \\ -\delta, & \text{if } x \in (b\delta, +\infty) \end{cases}$$

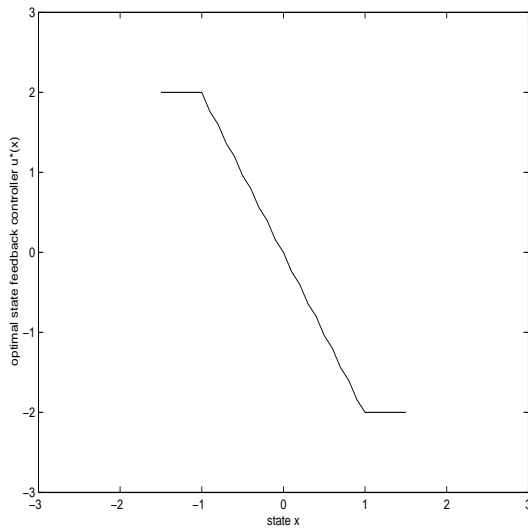
This controller is the same as that in Example 1 of [18]. □



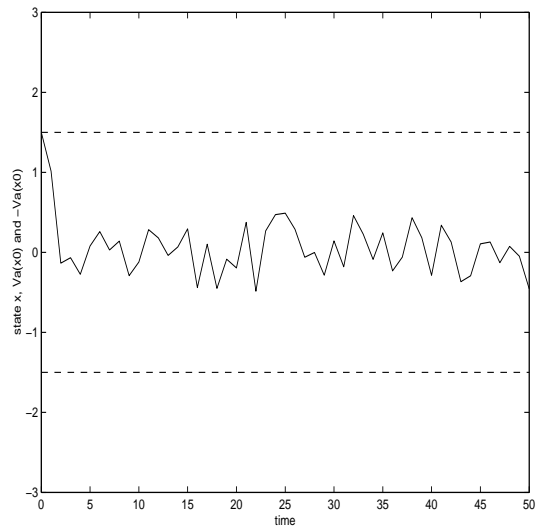
(a) Value function V_a . Note that when $x < -1.5$ or $x > 1.5$, $V_a(x) = +\infty$



(b) Trajectory of disturbance w .



(c) Optimal state feedback controller.



(d) State trajectory of close-loop system: $a_{0l} = -1.5, a_{0r} = 1.5, \beta = 1.5, x_0 = 1.5, V_a(x_0) = 1.5$.

Figure 5.1: Example 1 - system with linear dynamics - state feedback synthesis.

5.1.2 Measurement Feedback Synthesis

Assume that

$$\delta \geq \frac{ad}{b}; \quad \max\{|a_{ol}|, |a_{or}|\} \leq \frac{b\delta - d}{a - 1}; \quad a_{or} - a_{ol} \leq \frac{2(b\delta - ad)}{a(a - 1)}. \quad (5.5)$$

We set

$$X_0 = B_0 = [a_{ol}, a_{or}]. \quad (5.6)$$

Then the set-valued state estimate is an interval given by

$$X_k = [a_{kl}, a_{kr}], k \geq 0. \quad (5.7)$$

where

$$\begin{aligned} a_{(k+1)l} &= y_k + bu_k + (a - 1) \max\{a_{kl}, y_k - d\}, \\ a_{(k+1)r} &= y_k + bu_k + (a - 1) \min\{a_{kr}, y_k + d\}. \end{aligned} \quad (5.8)$$

This means that when X_0 is of the form (5.6), the results of Section 4.5 apply on a two-dimensional space $\hat{S} \subset \bar{S}'$.

The dynamic programming equation (4.58) was solved analytically for the value function $\hat{W}_a(X)$ and corresponding state feedback controller $\hat{\mathbf{u}}^*(X)$. Detailed calculations are provided in Appendix A.1.2. The value function is

$$\hat{W}_a(X) \triangleq W_a(\delta_X) = \hat{W}_a(a_l, a_r) = \max\{|a_l|, |a_r|, d + \frac{a}{2}(a_r - a_l)\}, \forall X = [a_l, a_r] \in \hat{S}$$

where

$$\hat{S} \triangleq \left\{ [a_l, a_r] : [a_l, a_r] \subset \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right], a_r - a_l \leq \frac{2(b\delta - ad)}{a(a - 1)} \right\},$$

and the corresponding optimal information state controller is given by

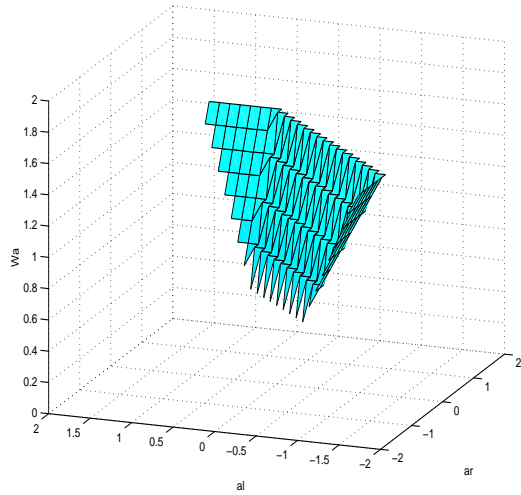
$$u_k = \hat{\mathbf{u}}^*(X_k) = \hat{\mathbf{u}}^*(a_{kl}, a_{kr}) \triangleq \begin{cases} \delta; & \text{if } (a_{kl} + a_{kr}) < -\frac{2b\delta}{a} \\ -\frac{a}{2b}(a_{kl} + a_{kr}); & \text{if } |a_{kl} + a_{kr}| \leq \frac{2b\delta}{a} \\ -\delta; & \text{if } (a_{kl} + a_{kr}) > \frac{2b\delta}{a} \end{cases} \quad (5.9)$$

The fact that this analytical solution $\hat{W}_a(a_l, a_r)$ of (4.58) is the value function is proven in Appendix A.1.2 (such equations could have multiple solutions).

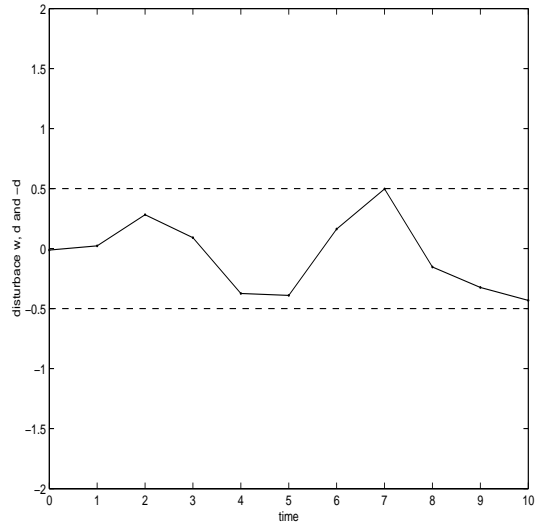
The dynamic programming equation (4.58) was also solved numerically ($a = 2, b = 1, d = 0.5, \delta = 2$). These numerical results correspond to the analytical solution above for the value function \hat{W}_a and corresponding controller. The results are shown in (a), (c) Figure 5.2.

Theorem 4.15 asserts that the closed-loop system should be LIB dissipative. Indeed, the calculations in Appendix A.1.2 verify LIB dissipation with respect to $B_0 = [a_{ol}, a_{or}]$ and $\beta = \max\{|a_{ol}|, |a_{or}|, \frac{a}{2}(a_{or} - a_{ol}) + d\}$, provided that the assumption (5.5) holds. This is illustrated in (b), (d) Figure 5.2 ($B_0 = [-0.5, 0.5], \beta = 1.5$).

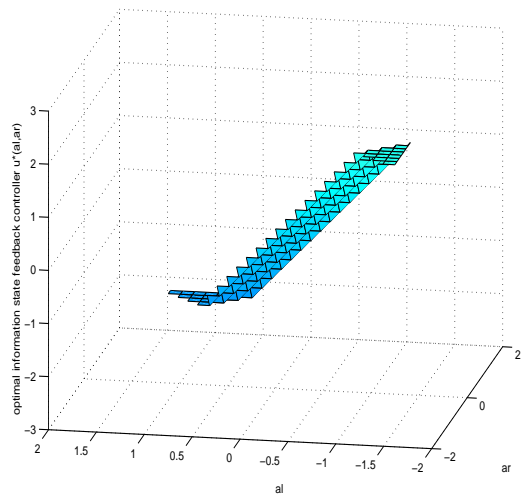
Remark 5.2 From (A.11), if $[a_{ol}, a_{or}] \notin \hat{S}$, then $\hat{W}_a(a_{ol}, a_{or}) = +\infty$, so the uniform LIB problem is not solvable when the assumption (5.5) does not hold. \square



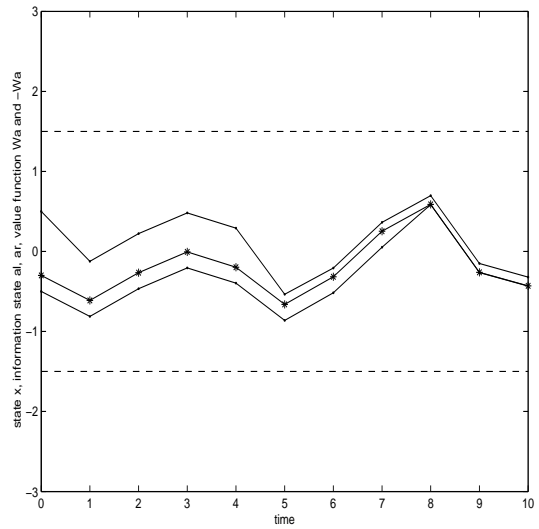
(a) Value function W_a . Note that when $a_l < -1.5$ or $a_r > 1.5$ or $a_r - a_l > 1$, $W_a(a_l, a_r) = +\infty$



(b) Trajectory of disturbance w .



(c) Optimal measurement feedback controller.



(d) State trajectory of close-loop system: $a_{0l} = -0.5, a_{0r} = 0.5, \beta = 1.5, x_0 = -0.3$.

Figure 5.2: Example 1 - system with linear dynamics - measurement feedback synthesis.

Remark 5.3 In contrast to the H_∞ problem, the certainty equivalence principle [1], [14], [10] can not be usefully applied for the linear system given in this example. Indeed, $W_0(p) = \langle p + V_a \rangle$ (where V_a is the state feedback value function defined by (5.3)) only satisfies the dynamic programming equation (A.12) on a very special set $\tilde{S} = \{[a_l, a_r] : a_l = a_r \in [-\frac{b\delta-d}{a-1}, \frac{b\delta-d}{a-1}]\}$. The set \tilde{S} is too small to be used for the measurement feedback synthesis. In fact, for any $0 < \varepsilon \leq \frac{b\delta-ad}{a(a-1)}$, $W_0([- \varepsilon, \varepsilon]) = \max\{\varepsilon, d\} < \hat{W}_a([- \varepsilon, \varepsilon]) = \max\{\varepsilon, a\varepsilon + d\}$, so it is impossible for $W_0([- \varepsilon, \varepsilon])$ to be an measurement feedback value function. \square

5.2 Example 2 - A System with Bilinear Dynamics

Consider one-dimensional system with bilinear dynamics

$$\begin{cases} \xi_{k+1} = \xi_k + b_1 \xi_k u_k + b_2 u_k + w_k \\ z_k = |\xi_k| \\ y_k = \xi_k + w_k \end{cases} \quad (5.10)$$

where $\xi_k, w_k, u_k, y_k, z_k \in \mathbf{R}, b_1 > 0, b_2 > 0$. We again consider the problem of uniform LIB synthesis. We suppose $B_0 = [a_{0l}, a_{0r}] \subset \mathbf{R}, \mathbf{W} = [-d, d], d > 0, \mathbf{U} = [-\delta, \delta], \delta > 0$. Assume that

$$\delta > \frac{1}{b_1}; \quad d \leq \frac{b_2 \delta}{b_1 \delta + 1}. \quad (5.11)$$

5.2.1 State Feedback Synthesis

We do not have an analytical solution to the dynamic programming equation (3.7). However, we obtain some estimates as follows.

An estimate of the value function is

$$\max\{|x_0|, d\} \leq V_a(x_0) \leq \begin{cases} \max\{|x_0|, \frac{b_2}{b_1} + d\}; & \text{if } x_0 \in (\frac{-b_2 \delta}{b_1 \delta - 1}, \frac{-b_2 \delta}{b_1 \delta + 1}) \\ \max\{|x_0|, d\}; & \text{if } x_0 \notin (\frac{-b_2 \delta}{b_1 \delta - 1}, \frac{-b_2 \delta}{b_1 \delta + 1}) \end{cases} \quad (5.12)$$

Assume that an optimal state feedback controller exists. Then with the optimal controller, the closed-loop system is uniform LIB dissipative with respect to B_0 and

$$\beta = \begin{cases} \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}; & \text{if } B_0 \cap (\frac{-b_2 \delta}{b_1 \delta - 1}, \frac{-b_2 \delta}{b_1 \delta + 1}) \neq \emptyset \\ \max\{|a_{0l}|, |a_{0r}|, d\}; & \text{if } B_0 \cap (\frac{-b_2 \delta}{b_1 \delta - 1}, \frac{-b_2 \delta}{b_1 \delta + 1}) = \emptyset \end{cases}$$

This is explained in Appendix A.2.1.

The dynamic programming equation (3.7) was solved numerically ($b_1 = 1, b_2 = 1, d = 0.5, \delta = 2$). These numerical results are consistent with the above estimates. The results are shown in (a), (c) Figure 5.3.

A simulation of the closed loop system is illustrated in (c), (d) Figure 5.3 ($b_1 = 1, b_2 = 1, d = 0.5, \delta = 2$), consistent with LIB dissipation with respect to $B_0 = [-1.2, 1.2]$ and $\beta = 1.5$ ($x_0 = -1.2, V_a(x_0) = 1.5$).

Remark 5.4 In the special case $\mathbf{U} = \mathbf{R}$, i.e. $\delta = +\infty$, under the assumption that

$$d \leq \frac{b_2}{b_1}, \quad (5.13)$$

we can prove that

$$V_a(x_0) = \begin{cases} \frac{b_2}{b_1} + d; & \text{if } x_0 = -\frac{b_2}{b_1} \\ \max\{|x_0|, d\}; & \text{if } x_0 \neq -\frac{b_2}{b_1} \end{cases}$$

with corresponding optimal state feedback controller

$$\mathbf{u}^*(x) = \begin{cases} \text{arbitrary}; & \text{if } x = -\frac{b_2}{b_1} \\ \frac{-x}{b_2 + b_1 x}; & \text{if } x \neq -\frac{b_2}{b_1} \end{cases} \quad (5.14)$$

With this controller the closed-loop system is uniform LIB dissipative with

$$\beta = \begin{cases} \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}; & \text{if } -\frac{b_2}{b_1} \in B_0 \\ \max\{|a_{0l}|, |a_{0r}|, d\}; & \text{if } -\frac{b_2}{b_1} \notin B_0 \end{cases}$$

□

5.2.2 Measurement Feedback Synthesis

As with Example 1, we can explicitly solve for the information state in terms of a set-valued state estimate, an interval.

Choose $X_0 = [a_{0l}, a_{0r}] = B_0$. The set-valued state estimate is given by

$$X_k = [a_{kl}, a_{kr}], k \geq 0. \quad (5.15)$$

where

(i) when $b_1 u_k \geq 0$,

$$\begin{aligned} a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k \max\{a_{kl}, y_k - d\}, \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k \min\{a_{kr}, y_k + d\}. \end{aligned}$$

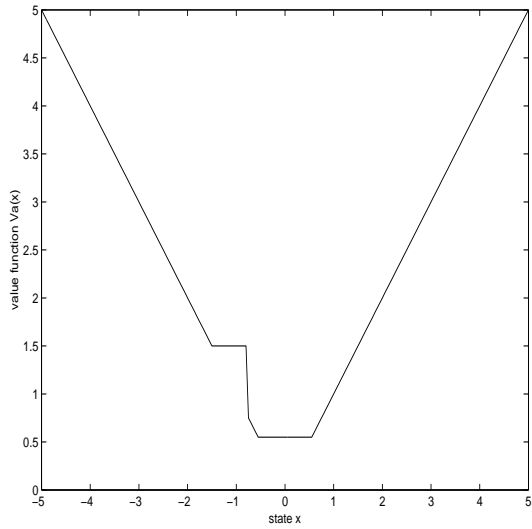
(ii) when $b_1 u_k < 0$

$$\begin{aligned} a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k \min\{a_{kr}, y_k + d\}, \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k \max\{a_{kl}, y_k - d\}. \end{aligned}$$

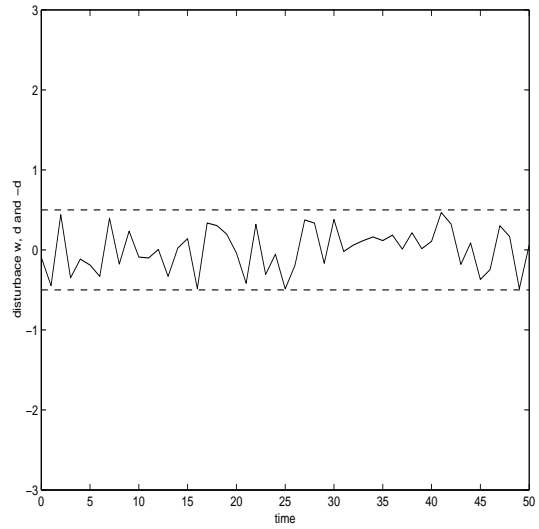
We do not have an analytical solution to the dynamic programming equation (4.58). However, we obtain some estimates as follows.

An estimate of the value function is

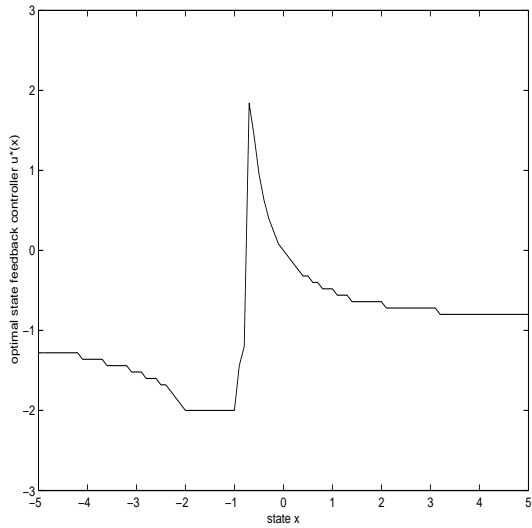
$$\max\{|a_l|, |a_r|, d\} \leq \hat{W}_a(X) = \hat{W}_a(a_l, a_r) \leq \max\{|a_l|, |a_r|, \frac{b_2}{b_1} + d\} \quad (5.16)$$



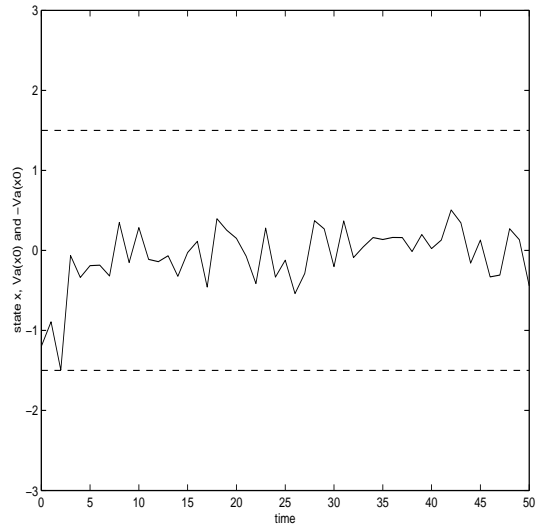
(a) Value function V_a .



(b) Trajectory of disturbance w .



(c) Optimal state feedback controller.



(d) State trajectory of close-loop system: $a_{0l} = -1.2, a_{0r} = 1.2, \beta = 1.5, x_0 = -1.2, V_a(x_0) = 1.5$.

Figure 5.3: Example 2 - system with bilinear dynamics - state feedback synthesis.

Assume that an optimal information state controller exists, then using the optimal controller, the closed-loop system is uniform LIB dissipative with respect to B_0 and

$$\beta = \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}.$$

The dynamic programming equation (4.58) was solved numerically ($b_1 = 1, b_2 = 1, d = 0.5, \delta = 2$). The numerical results are consistent with the above estimates. The results are shown in (a), (c) Figure 5.4.

A simulation of the closed loop system is illustrated in (b), (d) Figure 5.4, consistent with LIB dissipation with respect to $B_0 = [-2, 2], \beta = 2$.

Acknowledgements. We wish to thanks M.C. Smith and I.R. Petersen for helpful discussions and indicating to us several useful references.

A Calculations for the Examples

A.1 Example 1

A.1.1 State Feedback

We demonstrate that the value function V_a is given by

$$V_a(x_0) = \max\{|x_0|, d\}, \quad \forall x_0 \in \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right]. \quad (\text{A.1})$$

Now for fixed ξ_k ,

$$\begin{aligned} \xi_{k+1} &= a\xi_k + bu_k + w_k \\ a\xi_k + bu_k - d &\leq \xi_{k+1} \leq a\xi_k + bu_k + d \\ |\xi_{k+1}| &\leq \max\{|bu_k + a\xi_k - d|, |bu_k + a\xi_k + d|\} \\ \sup_{w_k \in \mathbf{W}} |\xi_{k+1}| &= \max\{|bu_k + a\xi_k - d|, |bu_k + a\xi_k + d|\} \end{aligned}$$

The choice

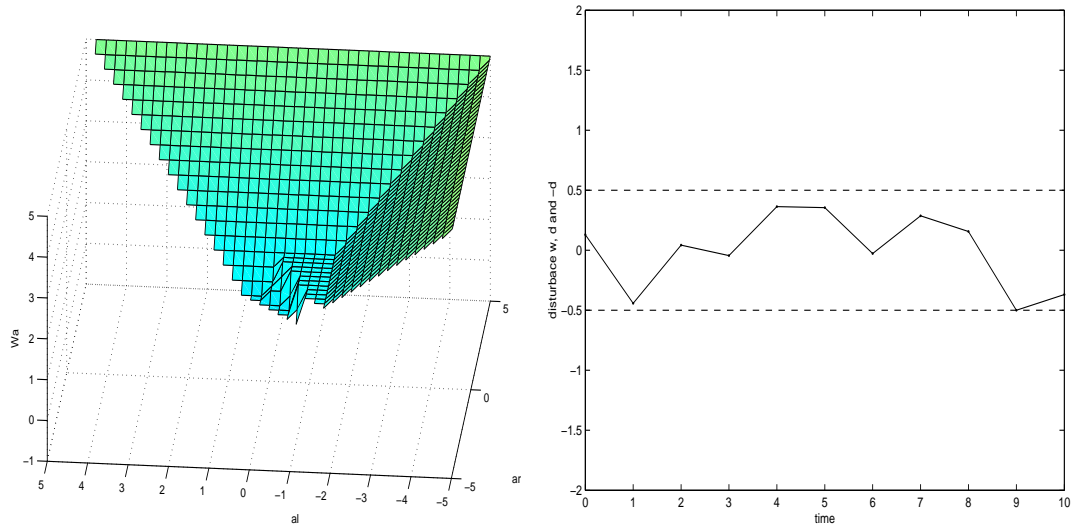
$$u_k = \mathbf{u}^*(\xi_k) = \begin{cases} \delta, & \text{if } \xi_k \in (-\infty, -\frac{b\delta}{a}) \\ -\frac{a}{b}\xi_k, & \text{if } \xi_k \in [-\frac{b\delta}{a}, \frac{b\delta}{a}] \\ -\delta, & \text{if } \xi_k \in (\frac{b\delta}{a}, +\infty) \end{cases}$$

attains the minimum

$$\inf_{u_k \in \mathbf{U}} \sup_{w_k \in \mathbf{W}} |\xi_{k+1}| = \begin{cases} -a\xi_k - b\delta + d, & \text{if } \xi_k \in (-\infty, -\frac{b\delta}{a}) \\ d, & \text{if } \xi_k \in [-\frac{b\delta}{a}, \frac{b\delta}{a}] \\ a\xi_k - b\delta + d, & \text{if } \xi_k \in (\frac{b\delta}{a}, +\infty) \end{cases}$$

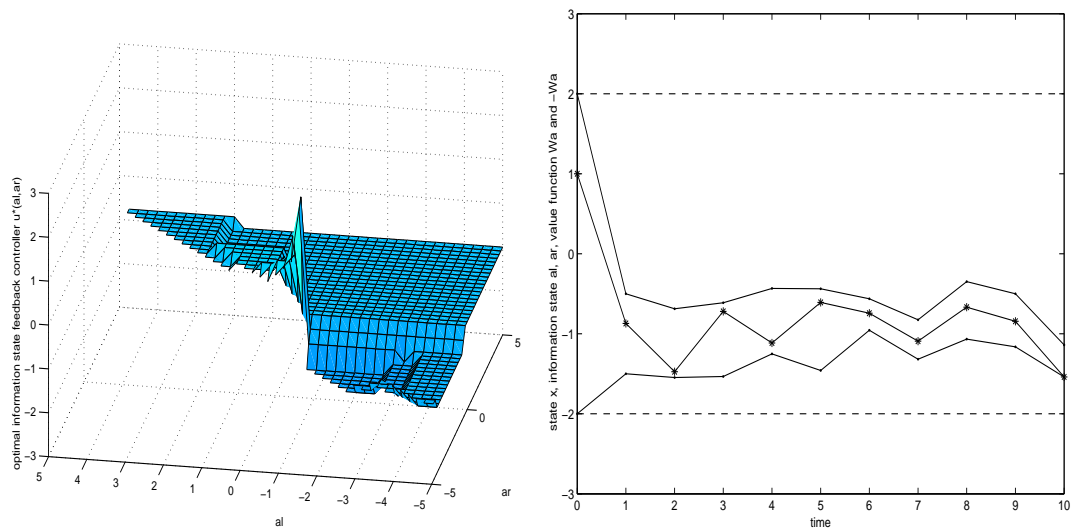
If the initial state $x_0 = \xi_0 \in [-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}]$, then using the controller

$$u_k = \mathbf{u}^*(\xi_k), \quad k \geq 0,$$



(a) Value function W_a .

(b) Trajectory of disturbance w .



(c) Optimal measurement feedback controller. (d) State and estimate trajectory of close-loop system: $a_{0l} = -2$, $a_{0r} = 2$, $\beta = 2$, $x_0 = 1$.

Figure 5.4: Example 2 - system with bilinear dynamics - measurement feedback synthesis.

we have

$$\xi_{k+1} \in \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right], \quad k \geq 0$$

and

$$\sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| = \max\{|x_0|, d\}.$$

Hence the value function has the upper bound:

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| \leq \max\{|x_0|, d\}, \quad \forall x_0 \in \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right]. \quad (\text{A.2})$$

Now we will prove that

$$V_a(x_0) \geq \max\{|x_0|, d\}, \quad \forall x_0 \in \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right]. \quad (\text{A.3})$$

Obviously, $V_a(x_0) \geq |\xi_0| = |x_0|$.

For any controller $K \in \mathcal{K}_{state}$ with the first step $u_0^K \in \mathbf{U}$,

$$\begin{aligned} & \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| : K \\ & \geq \sup_{w_0 \in \mathbf{W}} |\xi_1| : u_0^K \\ & \geq \inf_{u \in \mathbf{U}} \sup_{w_0 \in \mathbf{W}} |\xi_1| \\ & = \begin{cases} -a\xi_0 - b\delta + d, & \text{if } \xi_0 \in \left[-\frac{b\delta - d}{a - 1}, -\frac{b\delta}{a}\right) \\ d, & \text{if } \xi_0 \in \left[-\frac{b\delta}{a}, \frac{b\delta}{a}\right] \\ a\xi_0 - b\delta + d, & \text{if } \xi_0 \in \left(\frac{b\delta}{a}, \frac{b\delta - d}{a - 1}\right] \end{cases} \\ & \geq d. \end{aligned}$$

Since $K \in \mathcal{K}_{state}$ is arbitrary,

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| \geq d$$

and (A.3) holds.

Thus (A.1) follows from (A.2) and (A.3).

When $x_0 > \frac{b\delta - d}{a - 1}$, $\forall K \in \mathcal{K}_{state}$, choose $w_k = d, k \geq 0$, since $u_k^K \in \mathbf{U}$, we have

$$\begin{aligned} \xi_1 &> a\xi_0 - b\delta + d > \xi_0 = x_0, \\ \xi_2 &> a\xi_1 - b\delta + d, \\ &\dots\dots \end{aligned}$$

and hence

$$\begin{aligned} \xi_1 - \frac{b\delta - d}{a - 1} &> a\xi_0 - b\delta + d - \frac{b\delta - d}{a - 1} = a\left(\xi_0 - \frac{b\delta - d}{a - 1}\right), \\ \xi_2 - \frac{b\delta - d}{a - 1} &> a\left(\xi_1 - \frac{b\delta - d}{a - 1}\right) > a^2\left(\xi_0 - \frac{b\delta - d}{a - 1}\right), \\ &\dots\dots \end{aligned}$$

Since $a > 1$, we have

$$\sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| = +\infty.$$

Since $K \in \mathcal{K}_{state}$ is arbitrary, we have

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| = +\infty.$$

Similarly, when $x_0 < -\frac{b\delta-d}{a-1}$, we also have

$$V_a(x_0) = +\infty.$$

Given the above, we have proved that

$$\text{dom}V_a = \left[-\frac{b\delta-d}{a-1}, \frac{b\delta-d}{a-1}\right]$$

and

$$V_a(x_0) = \max\{|x_0|, d\}, \quad \forall x_0 \in \text{dom}V_a.$$

We now check that the value function satisfies the dynamic programming equation

$$V_a(x) = \max\{|x|, \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(f(x, u, w))\}, \quad \forall x \in \text{dom}V_a. \quad (\text{A.4})$$

In fact, $\forall x \in \text{dom}V_a$, $V_a(x) = \max\{|x|, d\}$.

For $u \in \mathbf{U}$ such that $\forall w \in \mathbf{W}$, $f(x, u, w) \in \text{dom}V_a$, we have

$$V_a(f(x, u, w)) = \max\{|f(x, u, w)|, d\}, \quad \forall w \in \mathbf{W}.$$

Since $f(x, u, w) = ax + bu + w$, the controller (5.4) satisfies

$$\sup_{w \in \mathbf{W}} V_a(f(x, \mathbf{u}^*(x), w)) = \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(f(x, u, w)) = \begin{cases} -ax - b\delta + d, & \text{if } x \in \left[-\frac{b\delta-d}{a-1}, -\frac{b\delta}{a}\right) \\ d, & \text{if } x \in \left[-\frac{b\delta}{a}, \frac{b\delta}{a}\right] \\ ax - b\delta + d, & \text{if } x \in \left(\frac{b\delta}{a}, \frac{b\delta-d}{a-1}\right] \end{cases}$$

Hence

$$d \leq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(f(x, u, w)) \leq \max\{|x|, d\}$$

and the dynamic programming equation (A.4) holds.

When the initial state is $x_0 \in \text{dom}V_a$, the closed-loop system satisfies

$$z_k \leq V_a(x_0), \quad \forall k \geq 0.$$

Equation (5.2) implies $B_0 \subset \text{dom}V_a$, so the closed-loop system is uniform LIB dissipative with respect to B_0 and

$$\beta = \max_{x_0 \in B_0} V_a(x_0) = \max_{x_0 \in B_0} \max\{|x_0|, d\} = \max\{|a_{0l}|, |a_{0r}|, d\}.$$

A.1.2 Measurement Feedback

We verify the expressions for \hat{W}_a and $\hat{S} \subset \text{dom}\hat{W}_a$. By equation (4.54), $x \in X_{k+1}$ if and only if there exists $x' \in X_k$, $w \in \mathbf{W}$ such that $f(x', u_k, w) = x$ and $h(x', u_k, w) = y_k$, i.e.

$$\begin{cases} ax' + bu_k + w = x \\ x' + w = y_k \end{cases}$$

that is $ax' + bu_k + y_k - x' = x$, or $(a-1)x' = x - y_k - bu_k$, hence

$$\begin{cases} x' = \frac{x - y_k - bu_k}{a-1} \\ w = y_k - x' \end{cases}$$

Suppose $X_k = [a_{kl}, a_{kr}]$, then

$$\begin{aligned} x' &\in X_k \\ \Leftrightarrow x - y_k - bu_k &\in [(a-1)a_{kl}, (a-1)a_{kr}] \\ \Leftrightarrow x &\in [y_k + bu_k + (a-1)a_{kl}, y_k + bu_k + (a-1)a_{kr}]. \end{aligned}$$

and

$$\begin{aligned} w &\in \mathbf{W} \\ \Leftrightarrow y_k - x' &\in [-d, d] \\ \Leftrightarrow x' - y_k &\in [-d, d] \\ \Leftrightarrow x' &\in [y_k - d, y_k + d] \\ \Leftrightarrow x - y_k - bu_k &\in [(a-1)(y_k - d), (a-1)(y_k + d)] \\ \Leftrightarrow x &\in [y_k + bu_k + (a-1)(y_k - d), y_k + bu_k + (a-1)(y_k + d)]. \end{aligned}$$

Hence

$$X_{k+1} = [a_{(k+1)l}, a_{(k+1)r}]$$

where

$$\begin{aligned} a_{(k+1)l} &= y_k + bu_k + (a-1) \max\{a_{kl}, y_k - d\}, \\ a_{(k+1)r} &= y_k + bu_k + (a-1) \min\{a_{kr}, y_k + d\}. \end{aligned}$$

In order that $X_{k+1} \neq \emptyset$, or $a_{(k+1)l} \leq a_{(k+1)r}$, y_k must satisfy

$$a_{kl} - d \leq y_k \leq a_{kr} + d.$$

Since $g(x, u, w) = |x|$, the value function is

$$\hat{W}_a(X) = \hat{W}_a(a_l, a_r) = \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_0, k-1} \{\max\{|a_{kl}|, |a_{kr}|\}\} : X_0 = X, u = K(y) \quad (\text{A.5})$$

For fixed X_k , we compute

$$\inf_{u_k \in \mathbf{U}} \sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\}.$$

The range of y_k is

$$a_{kl} - d \leq y_k \leq a_{kr} + d$$

Consider four cases:

$$(Case 1) \quad a_{kl} - d \leq y_k \leq a_{kl} + d$$

$$\begin{aligned} y_k &\leq a_{kl} + d \\ y_k - d &\leq a_{kl} \\ a_{(k+1)l} &= y_k + bu_k + (a-1)a_{kl} \\ a_{kl} - d + bu_k + (a-1)a_{kl} &\leq a_{(k+1)l} \leq a_{kl} + d + bu_k + (a-1)a_{kl} \\ bu_k + aa_{kl} - d &\leq a_{(k+1)l} \leq bu_k + aa_{kl} + d \\ |a_{(k+1)l}| &\leq \max\{|bu_k + aa_{kl} - d|, |bu_k + aa_{kl} + d|\}. \end{aligned}$$

$$(Case 2) \quad a_{kl} + d \leq y_k \leq a_{kr} + d$$

$$\begin{aligned} y_k &\geq a_{kl} + d \\ y_k - d &\geq a_{kl} \\ a_{(k+1)l} &= y_k + bu_k + (a-1)(y_k - d) \\ a_{kl} + d + bu_k + (a-1)a_{kl} &\leq a_{(k+1)l} \leq a_{kr} + d + bu_k + (a-1)a_{kr} \\ bu_k + aa_{kl} + d &\leq a_{(k+1)l} \leq bu_k + aa_{kr} + d \\ |a_{(k+1)l}| &\leq \max\{|bu_k + aa_{kl} + d|, |bu_k + aa_{kr} + d|\}. \end{aligned}$$

$$(Case 3) \quad a_{kl} - d \leq y_k \leq a_{kr} - d$$

$$\begin{aligned} y_k &\leq a_{kr} - d \\ y_k + d &\leq a_{kr} \\ a_{(k+1)r} &= y_k + bu_k + (a-1)(y_k + d) \\ a_{kl} - d + bu_k + (a-1)a_{kl} &\leq a_{(k+1)r} \leq a_{kr} - d + bu_k + (a-1)a_{kr} \\ bu_k + aa_{kl} - d &\leq a_{(k+1)r} \leq bu_k + aa_{kr} - d \\ |a_{(k+1)r}| &\leq \max\{|bu_k + aa_{kl} - d|, |bu_k + aa_{kr} - d|\}. \end{aligned}$$

$$(Case 4) \quad a_{kr} - d \leq y_k \leq a_{kr} + d$$

$$\begin{aligned} y_k &\geq a_{kr} - d \\ y_k + d &\geq a_{kr} \\ a_{(k+1)r} &= y_k + bu_k + (a-1)a_{kr} \\ a_{kr} - d + bu_k + (a-1)a_{kr} &\leq a_{(k+1)r} \leq a_{kr} + d + bu_k + (a-1)a_{kr} \\ bu_k + aa_{kr} - d &\leq a_{(k+1)r} \leq bu_k + aa_{kr} + d \\ |a_{(k+1)r}| &\leq \max\{|bu_k + aa_{kr} - d|, |bu_k + aa_{kr} + d|\}. \end{aligned}$$

Next, from above

$$\begin{aligned} &\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} \\ &= \max\{|bu_k + aa_{kl} - d|, |bu_k + aa_{kl} + d|, |bu_k + aa_{kr} - d|, |bu_k + aa_{kr} + d|\}. \end{aligned}$$

The choice

$$u_k = \begin{cases} \delta; & \text{if } (a_{kl} + a_{kr}) < -\frac{2b\delta}{a} \\ -\frac{a}{2b}(a_{kl} + a_{kr}); & \text{if } |a_{kl} + a_{kr}| \leq \frac{2b\delta}{a} \\ -\delta; & \text{if } (a_{kl} + a_{kr}) > \frac{2b\delta}{a} \end{cases} \quad (\text{A.6})$$

attains the minimum

$$\inf_{u_k \in \mathbf{U}} \sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = \begin{cases} -aa_{kl} - b\delta + d; & \text{if } (a_{kl} + a_{kr}) < -\frac{2b\delta}{a} \\ \frac{a}{2}(a_{kr} - a_{kl}) + d; & \text{if } |a_{kl} + a_{kr}| \leq \frac{2b\delta}{a} \\ aa_{kr} - b\delta + d; & \text{if } (a_{kl} + a_{kr}) > \frac{2b\delta}{a} \end{cases}$$

Now we prove that

$$\hat{S} \subset \text{dom}\hat{W}_a, \quad (\text{A.7})$$

and

$$\hat{W}_a(a_l, a_r) = \max\{|a_l|, |a_r|, d + \frac{a}{2}(a_r - a_l)\}, \quad \forall [a_l, a_r] \in \hat{S}. \quad (\text{A.8})$$

where

$$\hat{S} \triangleq \left\{ [a_l, a_r] : [a_l, a_r] \subset \left[-\frac{b\delta - d}{a - 1}, \frac{b\delta - d}{a - 1}\right], a_r - a_l \leq \frac{2(b\delta - ad)}{a(a - 1)} \right\}. \quad (\text{A.9})$$

(Step 1) We first prove that \hat{S} is an invariant set under the u_k in (5.9), i.e. if $[a_{kl}, a_{kr}] \in \hat{S}$, then

$$\forall y_k, [a_{(k+1)l}, a_{(k+1)r}] \in \hat{S}.$$

If $(a_{kl} + a_{kr}) < -\frac{2b\delta}{a}$, then $u_k = \delta$,

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = -aa_{kl} - b\delta + d \leq a\frac{b\delta - d}{a - 1} - b\delta + d = \frac{b\delta - d}{a - 1}.$$

If $(a_{kl} + a_{kr}) > \frac{2b\delta}{a}$, then $u_k = -\delta$,

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = aa_{kr} - b\delta + d \leq a\frac{b\delta - d}{a - 1} - b\delta + d = \frac{b\delta - d}{a - 1}.$$

If $|a_{kl} + a_{kr}| \leq \frac{2b\delta}{a}$, then $u_k = -\frac{a}{2b}(a_{kl} + a_{kr})$,

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = \frac{a}{2}(a_{kr} - a_{kl}) + d \leq \frac{a}{2} \frac{2(b\delta - ad)}{a(a - 1)} + d = \frac{b\delta - ad}{a - 1} + d = \frac{b\delta - d}{a - 1}.$$

We have, from (5.8), $\forall u_k$,

$$\sup_{y_k} (a_{(k+1)r} - a_{(k+1)l}) \leq (a - 1) \min\{2d, (a_{kr} - a_{kl})\} \leq (a_{kr} - a_{kl}) \quad (\text{A.10})$$

Hence

$$a_{(k+1)r} - a_{(k+1)l} \leq \frac{2(b\delta - ad)}{a(a - 1)}.$$

and $[a_{(k+1)l}, a_{(k+1)r}] \in \hat{S}$.

(Step 2) Now we prove that if $[a_{0l}, a_{0r}] \in \hat{S}$, then using the u_k in (5.9),

$$\begin{aligned} & \sup_{k \geq 0} \sup_{y_{0,k-1}} \max\{|a_{kl}|, |a_{kr}|\} : X_0 = [a_{0l}, a_{0r}] \\ & \leq \max\{|a_{0l}|, |a_{0r}|, d + \frac{a}{2}(a_{0r} - a_{0l})\} \end{aligned}$$

and hence

$$\begin{aligned}\hat{W}_a(a_{0l}, a_{0r}) &= \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_{0,k-1}} \{\max\{|a_{kl}|, |a_{kr}|\} : X_0 = [a_{0l}, a_{0r}], u = K(y)\} \\ &\leq \max\{|a_{0l}|, |a_{0r}|, d + \frac{a}{2}(a_{0r} - a_{0l})\}.\end{aligned}$$

If $(a_{kl} + a_{kr}) < -\frac{2b\delta}{a}$, then $u_k = \delta$, since $-a_{kl} \leq \frac{b\delta-d}{a-1}$, we have

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = -aa_{kl} - b\delta + d \leq -a_{kl} = |a_{kl}|.$$

If $(a_{kl} + a_{kr}) > \frac{2b\delta}{a}$, then $u_k = -\delta$, since $a_{kr} \leq \frac{b\delta-d}{a-1}$, we have

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = aa_{kr} - b\delta + d \leq a_{kr} = |a_{kr}|.$$

If $|a_{kl} + a_{kr}| \leq \frac{2b\delta}{a}$, then $u_k = -\frac{a}{2b}(a_{kl} + a_{kr})$,

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = \frac{a}{2}(a_{kr} - a_{kl}) + d.$$

Using (A.10), we have

$$\sup_{k \geq 0} \sup_{y_{0,k-1}} \max\{|a_{kl}|, |a_{kr}|\} \leq \max\{|a_{0l}|, |a_{0r}|, d + \frac{a}{2}(a_{0r} - a_{0l})\}.$$

(Step 3) Now we prove that when $[a_{0l}, a_{0r}] \in \hat{S}$,

$$\begin{aligned}\hat{W}_a(a_{0l}, a_{0r}) &= \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_{0,k-1}} \{\max\{|a_{kl}|, |a_{kr}|\} : X_0 = [a_{0l}, a_{0r}], u = K(y)\} \\ &\geq \max\{|a_{0l}|, |a_{0r}|, d + \frac{a}{2}(a_{0r} - a_{0l})\}.\end{aligned}$$

Obviously,

$$\hat{W}_a(a_{0l}, a_{0r}) \geq \max\{|a_{0l}|, |a_{0r}|\}.$$

For any controller $K \in \mathcal{K}$ with the first step $u_0^K \in \mathbf{U}$,

$$\begin{aligned}&\sup_{k \geq 0} \sup_{y_{0,k-1}} \max\{|a_{kl}|, |a_{kr}|\} : K \\ &\geq \sup_{y_0} \max\{|a_{1l}|, |a_{1r}|\} : u_0^K \\ &\geq \inf_{u \in \mathbf{U}} \sup_{y_0} \max\{|a_{1l}|, |a_{1r}|\} \\ &= \begin{cases} -aa_{0l} - b\delta + d; & \text{if } (a_{0l} + a_{0r}) < -\frac{2b\delta}{a} \\ \frac{a}{2}(a_{0r} - a_{0l}) + d; & \text{if } |a_{0l} + a_{0r}| \leq \frac{2b\delta}{a} \\ aa_{0r} - b\delta + d; & \text{if } (a_{0l} + a_{0r}) > \frac{2b\delta}{a} \end{cases} \\ &\geq d + \frac{a}{2}(a_{0r} - a_{0l}).\end{aligned}$$

Since $K \in \mathcal{K}$ is arbitrary,

$$\hat{W}_a(a_{0l}, a_{0r}) = \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_{0,k-1}} \{\max\{|a_{kl}|, |a_{kr}|\} : X_0 = [a_{0l}, a_{0r}], u = K(y)\} \geq d + \frac{a}{2}(a_{0r} - a_{0l}).$$

(Step 4) Now we prove that

$$[a_{0l}, a_{0r}] \notin \hat{S} \Rightarrow \hat{W}_a(a_{0l}, a_{0r}) = +\infty. \quad (\text{A.11})$$

When $a_{0r} > \frac{b\delta-d}{a-1}$, $\forall K \in \mathcal{K}$, choose $y_k = a_{kr} + d, k \geq 0$, since $u_k^K \in \mathbf{U}$, we have

$$\begin{aligned} a_{1r} &> aa_{0r} - b\delta + d > a_{0r}, \\ a_{2r} &> aa_{1r} - b\delta + d, \\ &\dots\dots \end{aligned}$$

and hence

$$\begin{aligned} a_{1r} - \frac{b\delta-d}{a-1} &> aa_{0r} - b\delta + d - \frac{b\delta-d}{a-1} = a(a_{0r} - \frac{b\delta-d}{a-1}), \\ a_{2r} - \frac{b\delta-d}{a-1} &> a(a_{1r} - \frac{b\delta-d}{a-1}) > a^2(a_{0r} - \frac{b\delta-d}{a-1}), \\ &\dots\dots \end{aligned}$$

Since $a > 1$, we have

$$\sup_{k \geq 0} \sup_{y_{0,k-1}} |a_{kr}| = +\infty.$$

Since $K \in \mathcal{K}$ is arbitrary, we have

$$\hat{W}_a(a_{0l}, a_{0r}) \geq \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_{0,k-1}} |a_{kr}| = +\infty.$$

Similarly, when $a_{0l} < -\frac{b\delta-d}{a-1}$, we also have

$$\hat{W}_a(a_{0l}, a_{0r}) = +\infty.$$

When $a_{0r} - a_{0l} > \frac{2(b\delta-ad)}{a(a-1)}$, $\forall K \in \mathcal{K}$, choose $\hat{y}_0 = a_{0l} - d, \check{y}_0 = a_{0r} + d$, we have

$$\hat{a}_{1l} = \hat{a}_{1r} = aa_{0l} - d + bu_0^K, \quad \check{a}_{1l} = \check{a}_{1r} = aa_{0r} + d + bu_0^K.$$

Since

$$\check{a}_{1r} - \hat{a}_{1l} = a(a_{0r} - a_{0l}) + 2d > \frac{2(b\delta - ad)}{a-1} + 2d = \frac{2(b\delta - d)}{a-1},$$

we have either $\hat{a}_{1l} < -\frac{b\delta-d}{a-1}$ or $\check{a}_{1r} > \frac{b\delta-d}{a-1}$ and hence

$$\hat{W}_a(a_{0l}, a_{0r}) = +\infty.$$

Using Steps 1-4, we have proved the formula (A.8).

Now we check that $\hat{W}_a(a_l, a_r)$ is a solution of the dynamic programming equation

$$\hat{W}_a(a_l, a_r) = \max\{|a_l|, |a_r|, \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \hat{W}_a(\hat{F}(a_l, a_r, u, y))\}, \quad \forall X = [a_l, a_r] \in \hat{S}, \quad (\text{A.12})$$

where $\hat{F}(a_l, a_r, u, y) = \hat{F}(X, u, y)$ is defined in (4.54).

In fact, $\hat{W}_a(X) = \hat{W}_a(a_l, a_r) = \max\{|a_l|, |a_r|, d + \frac{a}{2}(a_r - a_l)\}$.

Since $\hat{F}(a_l, a_r, u, y) = [\tilde{a}_l, \tilde{a}_r]$ where

$$\begin{aligned}\tilde{a}_l &= y + bu + (a - 1) \max\{a_l, y - d\}, \\ \tilde{a}_r &= y + bu + (a - 1) \min\{a_r, y + d\}.\end{aligned}$$

$$\hat{W}_a(\hat{F}(a_l, a_r, u, y)) = \max\{|\tilde{a}_l|, |\tilde{a}_r|, d + \frac{a}{2}(\tilde{a}_r - \tilde{a}_l)\}.$$

The controller (5.9) satisfies

$$\begin{aligned}& \sup_{y \in \mathbf{Y}} \hat{W}_a(\hat{F}(a_l, a_r, \hat{\mathbf{u}}^*(a_l, a_r), y)) \\ &= \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \hat{W}_a(\hat{F}(a_l, a_r, u, y)) \\ &= \begin{cases} \max\{-aa_l - b\delta + d, d + \frac{a}{2}(\tilde{a}_r - \tilde{a}_l)\}; & \text{if } (a_l + a_r) < -\frac{2b\delta}{a} \\ \max\{\frac{a}{2}(a_r - a_l) + d, d + \frac{a}{2}(\tilde{a}_r - \tilde{a}_l)\}; & \text{if } |a_l + a_r| \leq \frac{2b\delta}{a} \\ \max\{aa_r - b\delta + d, d + \frac{a}{2}(\tilde{a}_r - \tilde{a}_l)\}; & \text{if } (a_l + a_r) > \frac{2b\delta}{a} \end{cases}\end{aligned}$$

Hence

$$\max\{|a_l|, |a_r|, d + \frac{a}{2}(a_r - a_l)\} \geq \inf_{u \in \mathbf{U}} \sup_{y \in \mathbf{Y}} \hat{W}_a(\hat{F}(a_l, a_r, u, y)) \geq \frac{a}{2}(a_r - a_l) + d$$

and the dynamic programming equation (A.12) holds.

Assumption (5.5) shows that $B_0 \in \hat{S} \subset \text{dom} \hat{W}_a$, so the closed-loop system is uniform LIB dissipative with respect to B_0 and $\beta = \hat{W}_a(X_0) = \hat{W}_a(a_{0l}, a_{0r}) = \max\{|a_{0l}|, |a_{0r}|, d + \frac{a}{2}(a_{0r} - a_{0l})\}$.

A.2 Example 2

A.2.1 State Feedback

We verify the bounds for V_a .

For fixed ξ_k ,

$$\begin{aligned}\xi_{k+1} &= \xi_k + b_1 \xi_k u_k + b_2 u_k + w_k \\ \xi_k + b_1 \xi_k u_k + b_2 u_k - d &\leq \xi_{k+1} \leq \xi_k + b_1 \xi_k u_k + b_2 u_k + d \\ \sup_{w_k \in \mathbf{W}} |\xi_{k+1}| &= \max\{|(b_2 + b_1 \xi_k)u_k + \xi_k - d|, |(b_2 + b_1 \xi_k)u_k + \xi_k + d|\} \\ \sup_{w_k \in \mathbf{W}} |\xi_{k+1}| &= \max\{|(b_2 + b_1 \xi_k)u_k + \xi_k - d|, |(b_2 + b_1 \xi_k)u_k + \xi_k + d|\}\end{aligned}$$

When $\xi_k \notin (\frac{-b_2\delta}{b_1\delta-1}, \frac{-b_2\delta}{b_1\delta+1})$, the choice

$$u_k^* = \frac{-\xi_k}{b_2 + b_1 \xi_k} \in \mathbf{U} \tag{A.13}$$

attains the minimum

$$\inf_{u_k \in \mathbf{U}} \sup_{w_k \in \mathbf{W}} |\xi_{k+1}| = d.$$

(i) When $\xi_0 = x_0 \notin (\frac{-b_2\delta}{b_1\delta-1}, \frac{-b_2\delta}{b_1\delta+1})$, we choose

$$u_0^* = \frac{-\xi_0}{b_2 + b_1\xi_0} \in \mathbf{U},$$

then we have

$$\xi_1 = w_0,$$

since

$$|w_0| \leq d \leq \frac{b_2\delta}{b_1\delta + 1},$$

we have

$$\xi_1 \notin (\frac{-b_2\delta}{b_1\delta - 1}, \frac{-b_2\delta}{b_1\delta + 1})$$

so we can use the controller

$$u_1^* = \frac{-\xi_1}{b_2 + b_1\xi_1} \in \mathbf{U}$$

and

$$\xi_2 \notin (\frac{-b_2\delta}{b_1\delta - 1}, \frac{-b_2\delta}{b_1\delta + 1}).$$

Generally, we can choose

$$u_k^* = \frac{-\xi_k}{b_2 + b_1\xi_k}, \forall k \geq 0$$

and under this controller,

$$\sup_{k \geq 0} \sup_{w_0, k-1 \in \mathcal{W}_{0, k-1}} |\xi_k| = \max\{|x_0|, d\}.$$

Hence the value function

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_0, k-1 \in \mathcal{W}_{0, k-1}} |\xi_k| \leq \max\{|x_0|, d\}.$$

(ii) When $\xi_0 \in (\frac{-b_2\delta}{b_1\delta-1}, \frac{-b_2\delta}{b_1\delta+1})$, the choice

$$u_k \equiv -\frac{1}{b_1} \in \mathbf{U}, \forall k \geq 0 \tag{A.14}$$

attains

$$\sup_{w_k \in \mathbf{W}} |\xi_{k+1}| = \frac{b_2}{b_1} + d.$$

Hence the value function

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_0, k-1 \in \mathcal{W}_{0, k-1}} |\xi_k| \leq \max\{|x_0|, \frac{b_2}{b_1} + d\}. \tag{A.15}$$

Now we will prove that

$$V_a(x_0) \geq \max\{|x_0|, d\}, \quad \forall x_0 \in \mathbf{R}. \tag{A.16}$$

Obviously, $V_a(x_0) \geq |x_0|$. For any controller $K \in \mathcal{K}_{state}$ with the first step u_0^K ,

$$\sup_{k \geq 0} \sup_{w_0, k-1 \in \mathcal{W}_0, k-1} |\xi_k| : K \geq \sup_{w_0 \in \mathbf{W}} |\xi_1| : u_0^K \geq \inf_{u \in \mathbf{U}} \sup_{w_0 \in \mathbf{W}} |\xi_1| \geq d.$$

Since $K \in \mathcal{K}_{state}$ is arbitrary,

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_0, k-1 \in \mathcal{W}_0, k-1} |\xi_k| \geq d.$$

Using the above, the estimation (5.12) holds.

Suppose there exists an optimal state feedback controller. Under this controller and with the initial state x_0 , the closed-loop system satisfies

$$z_k \leq V_a(x_0), \quad \forall k \geq 0.$$

If $B_0 \cap \left(\frac{-b_2\delta}{b_1\delta-1}, \frac{-b_2\delta}{b_1\delta+1}\right) = \emptyset$, the closed-loop system is uniform LIB dissipative with respect to B_0 and

$$\beta = \max_{x_0 \in B_0} V_a(x_0) = \max_{x_0 \in B_0} \max\{|x_0|, d\} = \max\{|a_{0l}|, |a_{0r}|, d\}.$$

If $B_0 \cap \left(\frac{-b_2\delta}{b_1\delta-1}, \frac{-b_2\delta}{b_1\delta+1}\right) \neq \emptyset$, the closed-loop system is uniform LIB dissipative with respect to B_0 and

$$\beta = \max_{x_0 \in B_0} V_a(x_0) \leq \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}.$$

When $\mathbf{U} = \mathbf{R}$

For any $\xi_k \in \mathbf{R}$, the choice

$$u_k^* = \begin{cases} \text{arbitrary}; & \text{if } b_2 + b_1\xi_k = 0 \\ \frac{-\xi_k}{b_2 + b_1\xi_k}; & \text{if } b_2 + b_1\xi_k \neq 0 \end{cases}$$

attains the minimum

$$\inf_{u_k \in \mathbf{U}} \sup_{w_k \in \mathbf{W}} |\xi_{k+1}| = \begin{cases} \frac{b_2}{b_1} + d; & \text{if } b_2 + b_1\xi_k = 0 \\ d; & \text{if } b_2 + b_1\xi_k \neq 0 \end{cases}$$

(i) When $\xi_0 = x_0$ satisfies $b_2 + b_1\xi_0 \neq 0$, we choose

$$u_0^* = \frac{-\xi_0}{b_2 + b_1\xi_0},$$

then we have

$$\xi_1 = w_0,$$

since

$$|w_0| \leq d < \frac{b_2}{b_1},$$

we have

$$b_2 + b_1\xi_1 \neq 0,$$

so we can use the controller

$$u_1^* = \frac{-\xi_1}{b_2 + b_1\xi_1},$$

and obtain

$$\xi_2 = w_1, \quad b_2 + b_1\xi_2 \neq 0,$$

generally, we can choose

$$u_k^* = \frac{-\xi_k}{b_2 + b_1\xi_k}, \quad \forall k \geq 0$$

and under this controller,

$$\sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| = \max\{|x_0|, d\}.$$

Hence the value function

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| \leq \max\{|x_0|, d\}.$$

Inequality (A.16) then implies

$$V_a(x_0) = \max\{|x_0|, d\}.$$

(ii) When $\xi_0 = x_0$ satisfies $b_2 + b_1\xi_0 = 0$, i.e. $x_0 = -\frac{b_2}{b_1}$, we can prove that

$$V_a(x_0) \geq \max\{|x_0|, \frac{b_2}{b_1} + d\}.$$

In fact, when $\xi_0 = x_0 = -\frac{b_2}{b_1}$, for any controller $K \in \mathcal{K}_{state}$ with the first step u_0^K ,

$$\sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| : K \geq \sup_{w_0 \in \mathbf{W}} |\xi_1| : u_0^K = \sup_{w_0 \in \mathbf{W}} \left| -\frac{b_2}{b_1} + w_0 \right| = \frac{b_2}{b_1} + d.$$

Since $K \in \mathcal{K}_{state}$ is arbitrary,

$$V_a(x_0) = \inf_{K \in \mathcal{K}_{state}} \sup_{k \geq 0} \sup_{w_{0,k-1} \in \mathcal{W}_{0,k-1}} |\xi_k| \geq \frac{b_2}{b_1} + d = \max\{|x_0|, \frac{b_2}{b_1} + d\}.$$

Using (A.15), when $\xi_0 = x_0 = -\frac{b_2}{b_1}$, we have

$$V_a(x_0) = \max\{|x_0|, \frac{b_2}{b_1} + d\} = \frac{b_2}{b_1} + d.$$

Thus we have proved that

$$V_a(x_0) = \begin{cases} \frac{b_2}{b_1} + d; & \text{if } x_0 = -\frac{b_2}{b_1} \\ \max\{|x_0|, d\}; & \text{if } x_0 \neq -\frac{b_2}{b_1} \end{cases}$$

Now we prove that the value function satisfies the dynamic programming equation

$$V_a(x) = \max\{|x|, \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(f(x, u, w))\}, \forall x \in \text{dom} V_a = \mathbf{R}.$$

PROOF. When $x \neq -\frac{b_2}{b_1}$, $V_a(x) = \max\{|x|, d\}$.

$$\forall u \in \mathbf{U}, \forall w \in \mathbf{W}, V_a(f(x, u, w)) = \begin{cases} \frac{b_2}{b_1} + d; & \text{if } f(x, u, w) = -\frac{b_2}{b_1} \\ \max\{|f(x, u, w)|, d\}; & \text{if } f(x, u, w) \neq -\frac{b_2}{b_1} \end{cases}$$

Since $f(x, u, w) = x + b_1xu + b_2u + w$, when $u^* = \frac{-x}{b_2+b_1x}$,

$$f(x, u^*, w) = w, \quad V_a(f(x, u^*, w)) = d,$$

hence

$$d \leq \inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(f(x, u, w)) \leq \sup_{w \in \mathbf{W}} V_a(f(x, u^*, w)) = d.$$

Hence the dynamic programming equation holds.

When $x = -\frac{b_2}{b_1}$, $V_a(x) = \frac{b_2}{b_1} + d$.

Since $\forall u \in \mathbf{U}, f(-\frac{b_2}{b_1}, u, w) = -\frac{b_2}{b_1} + w$ and

$$V_a(-\frac{b_2}{b_1} + w) = \begin{cases} \frac{b_2}{b_1} + d; & \text{if } w = 0 \\ \max\{|-\frac{b_2}{b_1} + w|, d\}; & \text{if } w \neq 0 \end{cases}$$

we have

$$\sup_{w \in \mathbf{W}} V_a(f(-\frac{b_2}{b_1}, u, w)) = \frac{b_2}{b_1} + d,$$

Since u is arbitrary, we have

$$\inf_{u \in \mathbf{U}} \sup_{w \in \mathbf{W}} V_a(f(-\frac{b_2}{b_1}, u, w)) = \frac{b_2}{b_1} + d.$$

Hence the dynamic programming equation also holds. □

When the initial state is x_0 , the closed-loop system satisfies

$$z_k \leq V_a(x_0), \forall k \geq 0.$$

If $-\frac{b_2}{b_1} \notin B_0$, the closed-loop system is uniform LIB dissipative with

$$\beta = \max_{x_0 \in B_0} V_a(x_0) = \max_{x_0 \in B_0} \max\{|x_0|, d\} = \max\{|a_{0l}|, |a_{0r}|, d\}.$$

If $-\frac{b_2}{b_1} \in B_0$, the closed-loop system is uniform LIB dissipative with

$$\beta = \max_{x_0 \in B_0} V_a(x_0) = \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}.$$

A.2.2 Measurement Feedback

Now we explain the estimates for \hat{W}_a . From (4.54), $x \in X_{k+1}$ if and only if there exists $x' \in X_k$, $w \in \mathbf{W}$ such that $f(x', u_k, w) = x$ and $h(x', u_k, w) = y_k$, i.e.

$$\begin{cases} x' + b_1 x' u_k + b_2 u_k + w = x \\ x' + w = y_k \end{cases}$$

that is $x' + b_1 x' u_k + b_2 u_k + y_k - x' = x$, or $b_1 u_k x' = x - y_k - b_2 u_k$.

When $b_1 u_k \neq 0$

$$\begin{cases} x' = \frac{x - y_k - b_2 u_k}{b_1 u_k} \\ w = y_k - x' = y_k - \frac{x - y_k - b_2 u_k}{b_1 u_k} \end{cases}$$

Suppose $X_k = [a_{kl}, a_{kr}]$, then

(1) when $b_1 u_k > 0$

$$\begin{aligned} x' &\in X_k \\ \Leftrightarrow x - y_k - b_2 u_k &\in [b_1 u_k a_{kl}, b_1 u_k a_{kr}] \\ \Leftrightarrow x &\in [y_k + b_2 u_k + b_1 u_k a_{kl}, y_k + b_2 u_k + b_1 u_k a_{kr}]. \end{aligned}$$

and

$$\begin{aligned} w &\in \mathbf{W} \\ \Leftrightarrow y_k - x' &\in [-d, d] \\ \Leftrightarrow x' - y_k &\in [-d, d] \\ \Leftrightarrow x' &\in [y_k - d, y_k + d] \\ \Leftrightarrow x - y_k - b_2 u_k &\in [b_1 u_k (y_k - d), b_1 u_k (y_k + d)] \\ \Leftrightarrow x &\in [y_k + b_2 u_k + b_1 u_k (y_k - d), y_k + b_2 u_k + b_1 u_k (y_k + d)]. \end{aligned}$$

Hence

$$X_{k+1} = [a_{(k+1)l}, a_{(k+1)r}] \tag{A.17}$$

where

$$\begin{aligned} a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k \max\{a_{kl}, y_k - d\}, \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k \min\{a_{kr}, y_k + d\}. \end{aligned}$$

(2) when $b_1 u_k < 0$

$$\begin{aligned} x' &\in X_k \\ \Leftrightarrow x - y_k - b_2 u_k &\in [b_1 u_k a_{kr}, b_1 u_k a_{kl}] \\ \Leftrightarrow x &\in [y_k + b_2 u_k + b_1 u_k a_{kr}, y_k + b_2 u_k + b_1 u_k a_{kl}]. \end{aligned}$$

and

$$\begin{aligned} w &\in \mathbf{W} \\ \Leftrightarrow x' &\in [y_k - d, y_k + d] \\ \Leftrightarrow x - y_k - b_2 u_k &\in [b_1 u_k (y_k + d), b_1 u_k (y_k - d)] \\ \Leftrightarrow x &\in [y_k + b_2 u_k + b_1 u_k (y_k + d), y_k + b_2 u_k + b_1 u_k (y_k - d)]. \end{aligned}$$

Hence

$$X_{k+1} = [a_{(k+1)l}, a_{(k+1)r}] \tag{A.18}$$

where

$$\begin{aligned} a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k \min\{a_{kr}, y_k + d\}, \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k \max\{a_{kl}, y_k - d\}. \end{aligned}$$

Remark A.1 When $b_1 u_k = 0$, $x = y_k + b_2 u_k$ and

$$X_{k+1} = [y_k + b_2 u_k, y_k + b_2 u_k]$$

can be regarded as a special case of the equations (A.17) or (A.18). □

Since $g(x, u, w) = |x|$, the definition of the value function is (A.5).

Now for fixed X_k , we compute

$$\inf_{u_k} \sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\}.$$

The range of y_k is

$$a_{kl} - d \leq y_k \leq a_{kr} + d$$

(1) when $b_1 u_k > 0$,

(Case 1) $a_{kl} - d \leq y_k \leq a_{kl} + d$

$$\begin{aligned} y_k &\leq a_{kl} + d \\ y_k - d &\leq a_{kl} \\ a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k a_{kl} \\ a_{kl} - d + b_2 u_k + b_1 u_k a_{kl} &\leq a_{(k+1)l} \leq a_{kl} + d + b_2 u_k + b_1 u_k a_{kl} \\ (b_2 + b_1 a_{kl})u_k + a_{kl} - d &\leq a_{(k+1)l} \leq (b_2 + b_1 a_{kl})u_k + a_{kl} + d \end{aligned}$$

$$|a_{(k+1)l}| \leq \max\{|(b_2 + b_1 a_{kl})u_k + a_{kl} - d|, |(b_2 + b_1 a_{kl})u_k + a_{kl} + d|\}.$$

(Case 2) $a_{kl} + d \leq y_k \leq a_{kr} + d$

$$\begin{aligned} y_k &\geq a_{kl} + d \\ y_k - d &\geq a_{kl} \\ a_{(k+1)l} &= y_k + b_2 u_k + b_1 u_k (y_k - d) \\ a_{kl} + d + b_2 u_k + b_1 u_k a_{kl} &\leq a_{(k+1)l} \leq a_{kr} + d + b_2 u_k + b_1 u_k a_{kr} \\ (b_2 + b_1 a_{kl})u_k + a_{kl} + d &\leq a_{(k+1)l} \leq (b_2 + b_1 a_{kr})u_k + a_{kr} + d \end{aligned}$$

$$|a_{(k+1)l}| \leq \max\{|(b_2 + b_1 a_{kl})u_k + a_{kl} + d|, |(b_2 + b_1 a_{kr})u_k + a_{kr} + d|\}.$$

(Case 3) $a_{kl} - d \leq y_k \leq a_{kr} - d$

$$\begin{aligned} y_k &\leq a_{kr} - d \\ y_k + d &\leq a_{kr} \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k (y_k + d) \\ a_{kl} - d + b_2 u_k + b_1 u_k a_{kl} &\leq a_{(k+1)r} \leq a_{kr} - d + b_2 u_k + b_1 u_k a_{kr} \\ (b_2 + b_1 a_{kl})u_k + a_{kl} - d &\leq a_{(k+1)r} \leq (b_2 + b_1 a_{kr})u_k + a_{kr} - d \end{aligned}$$

$$|a_{(k+1)r}| \leq \max\{|(b_2 + b_1 a_{kl})u_k + a_{kl} - d|, |(b_2 + b_1 a_{kr})u_k + a_{kr} - d|\}.$$

(Case 4) $a_{kr} - d \leq y_k \leq a_{kr} + d$

$$\begin{aligned} y_k &\geq a_{kr} - d \\ y_k + d &\geq a_{kr} \\ a_{(k+1)r} &= y_k + b_2 u_k + b_1 u_k a_{kr} \\ a_{kr} - d + b_2 u_k + b_1 u_k a_{kr} &\leq a_{(k+1)r} \leq a_{kr} + d + b_2 u_k + b_1 u_k a_{kr} \\ (b_2 + b_1 a_{kr})u_k + a_{kr} - d &\leq a_{(k+1)r} \leq (b_2 + b_1 a_{kr})u_k + a_{kr} + d \end{aligned}$$

$$|a_{(k+1)r}| \leq \max\{|(b_2 + b_1 a_{kr})u_k + a_{kr} - d|, |(b_2 + b_1 a_{kr})u_k + a_{kr} + d|\}.$$

From above

$$\begin{aligned} &\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} \\ &= \max\{|(b_2 + b_1 a_{kl})u_k + a_{kl} - d|, |(b_2 + b_1 a_{kl})u_k + a_{kl} + d|, \\ &\quad |(b_2 + b_1 a_{kr})u_k + a_{kr} - d|, |(b_2 + b_1 a_{kr})u_k + a_{kr} + d|\}. \end{aligned} \quad (\text{A.19})$$

Similarly, when $b_1 u_k < 0$, we also have (A.19).

Remark A.2 When $b_1 u_k = 0$,

$$\begin{aligned} &\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} \\ &= \sup_{y_k} |y_k + b_2 u_k| \\ &= \max\{|b_2 u_k + a_{kl} - d|, |b_2 u_k + a_{kr} + d|\} \\ &= \max\{|b_2 u_k + a_{kl} - d|, |b_2 u_k + a_{kl} + d|, |b_2 u_k + a_{kr} - d|, |b_2 u_k + a_{kr} + d|\} \end{aligned}$$

can be regarded as a special case of (A.19). \square

For any a_{kl}, a_{kr} , the choice

$$u_k \equiv -\frac{1}{b_1} \in \mathbf{U}, \forall k \geq 0$$

attains

$$\sup_{y_k} \max\{|a_{(k+1)l}|, |a_{(k+1)r}|\} = \frac{b_2}{b_1} + d.$$

Hence $\forall X = [a_l, a_r]$, the value function

$$\hat{W}_a(X) = \hat{W}_a(a_l, a_r) = \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_0, k-1} \max\{|a_{kl}|, |a_{kr}|\} \leq \max\{|a_l|, |a_r|, \frac{b_2}{b_1} + d\}. \quad (\text{A.20})$$

Now we prove that

$$\hat{W}_a(a_l, a_r) \geq \max\{|a_l|, |a_r|, d\}.$$

In fact, obviously, $\hat{W}_a(a_l, a_r) \geq \max\{|a_l|, |a_r|\}$. For any controller $K \in \mathcal{K}$ with the first step u_0^K ,

$$\begin{aligned}
& \sup_{k \geq 0} \sup_{y_{0,k-1}} \max\{|a_{kl}|, |a_{kr}|\} : K \\
& \geq \sup_{y_0} \max\{|a_{1l}|, |a_{1r}|\} : u_0^K \\
& \geq \inf_{u \in \mathbf{U}} \sup_{y_0} \max\{|a_{1l}|, |a_{1r}|\} \\
& = \inf_{u \in \mathbf{U}} \max\{|(b_2 + b_1 a_l)u + a_l - d|, |(b_2 + b_1 a_l)u + a_l + d|, \\
& \quad |(b_2 + b_1 a_r)u + a_r - d|, |(b_2 + b_1 a_r)u + a_r + d|\} \\
& \geq d.
\end{aligned}$$

Since $K \in \mathcal{K}$ is arbitrary,

$$\hat{W}_a(a_l, a_r) = \inf_{K \in \mathcal{K}} \sup_{k \geq 0} \sup_{y_{0,k-1}} \max\{|a_{kl}|, |a_{kr}|\} \geq d.$$

Thus the estimate (5.16) holds.

Now choose $X_0 = B_0 = [a_{0l}, a_{0r}]$ and use the controller (A.14), then the closed-loop system is uniform LIB dissipative with $\beta = \hat{W}(a_{0l}, a_{0r}) \leq \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}$.

Suppose an optimal information state controller exists, then the closed-loop system is also uniform LIB dissipative with $\beta = \hat{W}(a_{0l}, a_{0r}) \leq \max\{|a_{0l}|, |a_{0r}|, \frac{b_2}{b_1} + d\}$.

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