Measurement Feedback Controller Design to Achieve Input to State Stability

S. Huang, M.R. James, D. Nešić and P.M. Dower

Abstract— **An approach for design of measurement feedback controllers achieving input-to-state (ISS) stability properties is presented. A synthesis procedure based on dynamic programming is given. We make use of recently developed results on controller synthesis to achieve uniform l^∞ bound [6]. Our results make an important connection between the ISS literature and nonlinear H^∞ design methods.**

I. INTRODUCTION

Analysis and design of control systems with disturbances is one of the central topics in control engineering that is continuing to attract a lot of research interest in the context of nonlinear systems. This trend has been driven by several major breakthroughs over the past 15 years that occurred in nonlinear H^∞ control (e.g. [3], [21], [5]) and the input to state stability (ISS) related literature (e.g. [19], [16], [2]). These two approaches have been developed relatively independently of each other and they differ in stability properties that are considered, tools that are used and questions that are asked. Both approaches have their advantages and disadvantages but they both provide invaluable tools and insight into the problems of analysis and design of nonlinear control systems with disturbances.

Nonlinear H^∞ control has its roots in the areas of LQ control and linear H^∞ control. The main objective of this research has been to translate all linear H^∞ control results to a nonlinear setting. In this context, it is typical to model the plant and controller as nonlinear operators and to consider L^2 stability with a finite (linear) gain of the closed loop system, which comes from its linear tradition. Moreover, this literature often aims at designing controllers that achieve minimum (optimal) gains from disturbance inputs to plant outputs and, hence, controller design often requires a solution of an appropriate dynamic programming equation (DPE) or inequality (DPI). An advantage of this approach is that it can be applied to a very broad class of plants and its main drawback is the heavy computation required to solve DPE/DPI [5]. Nevertheless, the methodology is fundamental and provides useful conceptual insights.

This work was supported by the Australian Research Council.

Shoudong Huang is with the Centre of Excellence in Autonomous Systems, Faculty of Engineering, The University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia. sdhuang@eng.uts.edu.au

Matt James is with the Department of Engineering, Australian National University, Canberra, ACT 0200, Australia. matthew.james@anu.edu.au

Dragan Nešić and Peter Dower are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia. d.nesic@ee.mu.oz.au, p.dower@ee.mu.oz.au

On the other hand, ISS related literature builds on the tradition of stability of dynamical systems and Lyapunov theory. Research in this area has concentrated on finding appropriate nonlinear generalizations of different finite gain input-output stability properties that are more natural in the nonlinear context and fully compatible with Lyapunov theory. The plant is modelled as a dynamical system with disturbance inputs and the related stability properties usually make use of nonlinear gains. Majority of ISS related research has concentrated on presenting different equivalent characterizations of ISS like properties [18], [19], [2], proving appropriate small gains theorems [10] and applying the ISS like properties to analysis and controller design. This literature is usually not concerned with computing minimum disturbance gains and the main tool for applying these results are Lyapunov like functions that are very difficult to find. We are not aware of any results that provide a systematic procedure for controller design for general nonlinear systems that achieves different ISS like stability properties for the plant dynamics.

It is the purpose of this paper to exploit techniques typically used in nonlinear H^∞ control to address the problem of controller design with the goal of achieving the ISS property for the plant dynamics. In particular, we use recent results on uniform l^∞ bounded (ULIB) robustness [6] that extend nonlinear H^∞ techniques to an appropriate l^∞ robostness property. Our main results show that the controller design problem achieving ISS property for the plant dynamics can be solved by solving another ULIB problem for an auxiliary augmented plant. Important features of our approach are: (i) we need to fix the desired ISS gain and transients bound prior to controller design; (ii) admissible controllers we consider are causal operators and our solutions can be interpreted as a dynamical controller with an appropriate initialization; (iii) we achieve an ISS bound only for the plant dynamics and controller dynamics is not considered; (iv) we consider the measurement feedback problem; (v) our controllers are obtained via solutions of appropriate DPE/DPI and in general they are computationally very demanding.

This paper is an extension of our state feedback ISS synthesis result [8]. A range of other ISS-like properties can be dealt with using the same framework and can be found in a full version of this paper [9].

This paper is organized as follows. Preliminaries and notations are given in Section II. In Section III, we present a modified definition for ISS property. In Section IV, we state the measurement feedback synthesis problem considered in this paper. The problem is then transferred into ULIB synthesis problem in Section V. In Section VI, the dynamic programming results are presented using the existing ULIB results. An illustrate example is given in Section VII. Conclusion is presented in Section VIII.

II. Preliminaries

Sets of real numbers, nonnegative real numbers, integers and nonnegative integers are denoted respectively as $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$ and $\mathbf{Z}_+$. Moreover, we denote

$$\bar{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}, \quad \tilde{\mathbf{R}} := \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}. \quad (1)$$

Recall that a function $\gamma : \mathbf{R}_+ \to \mathbf{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a function of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s, \cdot)$ decreases to zero.

Sontag [15] proved the following lemma on $\mathcal{KL}$ functions that we need.

Lemma 2.1: [15] Given arbitrary $\beta \in \mathcal{KL}$, there exist two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\beta(s,t) \leq \beta_1(s,t) = \alpha_1\left(\alpha_2(s)e^{-t}\right), \quad \forall s \geq 0, t \geq 0 \;. \quad (2)$$

Given $\mathbf{W} \subseteq \mathbf{R}^s$, we use the following notation for signals:

$$\begin{array}{rcl} w_{[0,k-1]} & := & \{w_0, \cdots, w_{k-1}\}, \forall k \geq 0, \\ \mathcal{W}_{[0,k-1]} & := & \{w_{[0,k-1]} : w_i \in \mathbf{W}, 0 \leq i \leq k-1\}, \\ \mathcal{W}_{[0,\infty)} & := & \{w_{[0,\infty)} : w_i \in \mathbf{W}\}. \end{array} \quad (3)$$

Sometimes we use the notation $w = w_{[0,\infty)}$. We use the convention that $w_{[0,-1]} = \emptyset$. In the sequel, we use the notation $\mathcal{U}_{[0,\infty)}, y_{[0,k-1]}, \mathcal{Y}_{[0,k-1]}, \mathcal{Y}_{[0,\infty)}$, etc, which have meanings analogous to (3). We also use the following notation:

$$\|w_{[0,k-1]}\|_\infty := \max_{0 \leq i \leq k-1} |w_i|$$

where $|\cdot|$ is the Euclidean norm. To simplify the notation, for any two vectors x_1 and x_2, sometimes we also denote $(x_1^T \; x_2^T)^T$ as (x_1, x_2).

III. Input to State Stability

Consider the following nonlinear system

$$x_{k+1} = f(x_k, w_k), \quad k \geq 0 \quad (4)$$

where $x_k \in \mathbf{R}^n$ is the state, $w_k \in \mathbf{W} \subseteq \mathbf{R}^s$ is the input. We denote by $\phi(k, x_0, w_{[0,k-1]})$ the solution of the system at time k that starts from the initial condition x_0 and under the action of the input $w_{[0,k-1]}$. Sometimes we simply use ϕ_k or x_k to denote $\phi(k, x_0, w_{[0,k-1]})$.

The definition of Input to State Stability (ISS) was given in [14], [11]. Here we modify it slightly to accommodate the possibility of restricting the range of initial conditions and input values.

Definition 3.1: Let $B_0 \subseteq \mathbf{R}^n, \mathbf{W} \subseteq \mathbf{R}^s$, the system (4) is input to state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the trajectories of (4) satisfy:

$$|\phi(k, x_0, w_{[0,k-1]})| \leq \beta(|x_0|, k) + \gamma(\|w_{[0,k-1]}\|_\infty),$$

for all $x_0 \in B_0$, $w_{[0,k-1]} \in \mathcal{W}_{[0,k-1]}$, and $k \geq 0$.

(In the original definition, $B_0 = \mathbf{R}^n$ and $\mathbf{W} = \mathbf{R}^s$.)

Remark 3.2: By Lemma 2.1, any $\beta \in \mathcal{KL}$ has an upper bound of the form $\beta_1(s,t) = \alpha_1\left(\alpha_2(s)e^{-t}\right)$. Notice that β_1 itself is also a $\mathcal{KL}$ function, so the system (4) is ISS if and only if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that the trajectories of (4) satisfy:

$$|\phi(k, x_0, w_{[0,k-1]})| \leq \alpha_1\left(\alpha_2(|x_0|)e^{-k}\right) + \gamma(\|w_{[0,k-1]}\|_\infty), \quad (5)$$

for all $x_0 \in B_0$, $w_{[0,k-1]} \in \mathcal{W}_{[0,k-1]}$, and $k \geq 0$. Certainly, the bound $\alpha_1\left(\alpha_2(|x_0|)e^{-k}\right)$ may be not as tight as the bound $\beta(|x_0|, k)$ with $\beta \in \mathcal{KL}$. In this paper, we will only consider the case when $\mathcal{KL}$ function is of the form $\alpha_1\left(\alpha_2(s)e^{-t}\right)$.

We find it useful to restate Definition 3.1 since its new form is more suited for our paper. First, note that the inequality (5) in the ISS definition is:

$$|\phi(k, x_0, w_{[0,k-1]})| - \alpha_1\left(\alpha_2(|x_0|)e^{-k}\right) - \gamma(\|w_{[0,k-1]}\|_\infty) \leq 0,$$

for all $x_0 \in B_0, w_{[0,k-1]} \in \mathcal{W}_{[0,k-1]}$, and $k \geq 0$.

Now we define function $\rho : \mathbf{R}^n \times \mathbf{Z}_+ \to \mathbf{R}_+$, for every $k \in \mathbf{Z}_+$, functions $\psi_k : \mathcal{W}_{[0,k-1]} \to \mathbf{R}_+$, and function $G : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, by

$$\begin{array}{rcl} \rho(x_0, k) & := & \alpha_2(|x_0|)e^{-k}, \\ \psi_k(w_{[0,k-1]}) & := & \|w_{[0,k-1]}\|_\infty, \end{array} \quad (6)$$

and

$$G(\phi, \rho, \psi) := |\phi| - \alpha_1(\rho) - \gamma(\psi), \quad (7)$$

where $\gamma \in \mathcal{K}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. We use the convention that $\psi_k(\emptyset) = 0$ and note that since $w_{[0,-1]} = \emptyset$, we have that $\psi_0(w_{[0,-1]}) = 0$.

Now we can restate the definition of ISS as follows.

Definition 3.3: Let $B_0 \subseteq \mathbf{R}^n$ and $\mathbf{W} \subseteq \mathbf{R}^s$ be given. The system (4) is called input to state stable (ISS) if there exist $\gamma \in \mathcal{K}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that with $\rho(\cdot,\cdot), \psi_k(\cdot)$ and $G(\cdot,\cdot,\cdot)$ defined by (6) and (7), we have that the solutions of the system (4) satisfy:

$$G(\phi(k, x_0, w_{[0,k-1]}), \rho(x_0, k), \psi_k(w_{[0,k-1]})) \leq 0, \quad (8)$$

for all $x_0 \in B_0, w_{[0,k-1]} \in \mathcal{W}_{[0,k-1]}, k \geq 0$.

Remark 3.4: The reason for restating Definition 3.1 as in Definition 3.3 is that the inequality (8) will be shown to be related to an inequality in the Uniform l^∞ Boundedness (ULIB) problem that was recently considered and solved in the literature [6].

Remark 3.5: A range of other stability and detectability properties can be captured by using the same Definition 3.3 in an appropriate manner by redefining the functions ρ, ψ_k, G etc. for each of the property. Such as integral input to state stability (iISS) [2], integral input to integral state stability (iIiSS) [15], input to output stability (IOS) [20], input output to state stability (IOSS) [12] and incremental input to state stability (δISS) [1], etc. Some details are provided in [9].

IV. Problem Statement

In this section we pose the measurement feedback problem that achieves ISS property for the plant state in the closed loop system.

A. Measurement Feedback ISS

Consider the nonlinear discrete-time system

$$\begin{array}{rcl} x_{k+1} & = & f(x_k, u_k, w_k),\ k \geq 0, \\ y_k & = & h(x_k, w_k),\ k \geq 0 \end{array} \tag{9}$$

Here $x_k \in \mathbf{R}^n, u_k \in \mathbf{U} \subseteq \mathbf{R}^m, w_k \in \mathbf{W} \subseteq \mathbf{R}^s, y_k \in \mathbf{R}^p$ are the state, control input, disturbance, and measured output, respectively.

Before we state the problem, we define the class of admissible controllers that our design will yield. For system (9), let $\mathbf{Y} = \text{range}\{h\} \subseteq \mathbf{R}^p$ and $\mathbf{U} \subseteq \mathbf{R}^m$ be given, define $\mathcal{Y}_{[0,\infty)}$ and $\mathcal{U}_{[0,\infty)}$ similarly as in (3). An *admissible* measurement feedback controller is a causal map $K : \mathcal{Y}_{[0,\infty)} \to \mathcal{U}_{[0,\infty)}$, meaning that for each time $k > 0$ if $y^1, y^2 \in \mathcal{Y}_{[0,\infty)}$ and $y^1_l = y^2_l$ for all $0 \leq l \leq k-1$ then $K(y^1)_k = K(y^2)_k$, i.e., the control at time k is independent of current and future measurements. We denote the set of admissible measurement feedback controllers as

$$\mathcal{C}_{mf} := \{K : \mathcal{Y}_{[0,\infty)} \to \mathcal{U}_{[0,\infty)}, K \text{ is causal}\}. \tag{10}$$

We sometimes abuse notation by writing $u_k = K(y_{[0,k-1]})$. Also, we still denote the trajectories of the plant in the closed loop system consisting of the system (9) and a given admissible controller $u_k = K(y_{[0,k-1]})$ as $\phi(k, x_0, u, w_{[0,k-1]})$.

Note that the class of admissible controllers is very large and it includes static and dynamic controllers, as well as a number of other configurations.

The problem that we consider is stated next.

Measurement Feedback ISS (MFISS) Problem: Consider system (9), let $B_0 \subseteq \mathbf{R}^n, \mathbf{W} \subseteq \mathbf{R}^s, \gamma \in \mathcal{K}, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ be given and define the functions $\rho(\cdot,\cdot), \psi_k(\cdot)$, and $G(\cdot,\cdot,\cdot)$ by (6) and (7). Find, if possible, an admissible measurement feedback controller $K \in \mathcal{C}_{mf}$ such that the trajectories of the plant in the closed loop system satisfy

$$G(\phi(k, x_0, u, w_{[0,k-1]}), \rho(x_0, k), \psi_k(w_{[0,k-1]})) \leq 0, \tag{11}$$

for all $x_0 \in B_0, w_{[0,k-1]} \in \mathcal{W}_{[0,k-1]}, k \geq 0$. When there exists such a controller, we say that the MFISS Problem is solvable for system (9).

Remark 4.1: Note a crucial difference between Definition 3.3 and the statement of the MFISS Problem. In the definition, we say that the property holds if *there exist* functions $\gamma \in \mathcal{K}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that the ISS inequality holds. However, in the statement of the MFISS Problem we *fix* all the functions $\gamma \in \mathcal{K}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and then attempt to find a controller that satisfies (11). Finding a design technique that does not require *a priori* fixing of the gain functions is highly desirable and is left for future research.

Remark 4.2: The MFISS problems require only that a desired bound is achieved on the solutions of the plant whereas no such requirement is imposed on the states of a possibly dynamic controller. There are three reasons for this: (i) ISS property for nonlinear systems provide a desired bound for any initial state of the system. However, for a closed-loop system, the initial state of the plant and the initial state of the controller play different roles. The initial state of the plant may be arbitrary. But the initial state of the controller can be chosen by the designer. Hence it may be too strong to require ISS bound to be obtained for any initial state of the plant *and any initial state of the controller* in the closed-loop system. (ii) We consider possibly dynamic feedback controller design where the dimension of the controller is not given before the design. (iii) This requirement is compatible with definitions of nonlinear H^∞ problems ([5]) and the ULIB problems that are stated next.

B. Uniform ℓ^∞ Bounded (ULIB) Problem

We shall show in section V that the MFISS Problem for the system (9) can be solved by solving the following controller synthesis problem for certain auxiliary systems. We first state the problem itself and then introduce the auxiliary systems in the following section.

Consider the following system

$$\begin{array}{rcl} x_{k+1} & = & f(x_k, u_k, w_k),\ k \geq 0, \\ y_k & = & h(x_k, w_k),\ k \geq 0, \\ z_k & = & g(x_k),\ k \geq 0. \end{array} \tag{12}$$

where $x_k \in \mathbf{R}^n, u_k \in \mathbf{U} \subseteq \mathbf{R}^m, w_k \in \mathbf{W} \subseteq \mathbf{R}^s, y_k \in \mathbf{R}^p$ are the state, control input, disturbance, and measured output, respectively. $z_k \in \mathbf{R}$ is the performance output quantity.

Notice that the dimensions of the states, the measurement outputs and the control inputs of system (9) and system (12) are all the same. So we still use the same notation $\mathcal{C}_{mf}$ and $\phi(k, x_0, u, w_{[0,k-1]})$ as those in the MFISS problem

Measurement Feedback ULIB (MFULIB) Problem: Consider system (12) and let $B_0 \subseteq \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ be given. Find, if possible, an admissible measurement feedback controller $K \in \mathcal{C}_{mf}$ such that the trajectories of the closed-loop system consisting of the plant (12) and the controller $K(\cdot)$ satisfy

$$g(\phi(k, x_0, u, w_{[0,k-1]})) \leq \lambda, \tag{13}$$

for all $x_0 \in B_0, w_{[0,k-1]} \in \mathcal{W}_{[0,k-1]}$, and $k \geq 0$. When there exists such a controller, we say that the MFULIB Problem is solvable for system (12).

Remark 4.3: When the trajectories of the closed-loop system satisfy (13), we say that the closed-loop system is uniform l^∞-bounded (ULIB) dissipative with respect to B_0 and λ. We emphasize that the solutions to the MFULIB Problem have been already obtained in [6].

Remark 4.4: Note the similarity between the bounds in (11) and (13) that are respectively used to define the MFISS and MFULIB problems. The main difference is that the bound in (11) depends directly on $\phi(k, x_0, u, w_{[0,k-1]})$, $\rho(x_0, k)$ and $\psi_k(w_{[0,k-1]})$ whereas the bound in (13) depends only on $\phi(k, x_0, u, w_{[0,k-1]})$. However, we will show in the next section that $\rho(x_0, k)$ and $\psi_k(w_{[0,k-1]})$ can be generated as solutions of auxiliary difference equations that are appropriately initialized and, moreover, we can solve the MFISS Problem for the system (9) by solving appropriate ULIB problems for augmented auxiliary systems that is appropriately initialized.

V. Problem Transformation

In this section we show how the MFISS Problem for the system (9) can be converted into appropriate MFULIB problem for auxiliary augmented systems.

Let $B_0 \subseteq \mathbf{R}^n$, $\mathbf{W} \subseteq \mathbf{R}^s$, $\gamma \in \mathcal{K}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ be given. For system (9), we define the following auxiliary system

$$\left\{ \begin{array}{rcl} x_{k+1} &=& f(x_k, u_k, w_k), \\ \zeta_{k+1} &=& e^{-1}\zeta_k, \\ \eta_{k+1} &=& \max\{\eta_k, |w_k|\}, \\ z_k &=& |x_k| - \alpha_1(\zeta_k) - \gamma(\eta_k), \\ y_k &=& h(x_k, w_k). \end{array} \right. \tag{14}$$

We also let:

$$\tilde{B}_0 = \left\{ \begin{pmatrix} x_0 \\ \alpha_2(|x_0|) \\ 0 \end{pmatrix} : x_0 \in B_0 \right\}, \quad \lambda = 0. \tag{15}$$

The following theorem shows a relationship of the MFISS Problem for system (9) and the MFULIB Problem for auxiliary system (14) with $\tilde{B}_0$ and λ defined in (15). The proof is omitted because of the space limitation.

Theorem 5.1: Let $\mathbf{Y} = \text{range}\{h\} \subseteq \mathbf{R}^p$ and $\mathbf{U} \subseteq \mathbf{R}^m$ be given and define the set of admissible controller $\mathcal{C}_{mf}$ as in (10). Let $B_0 \subseteq \mathbf{R}^n$, $\mathbf{W} \subseteq \mathbf{R}^s$, $\gamma \in \mathcal{K}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ be given. Then, the following statements are equivalent:

(i) The MFISS Problem is solvable for system (9).
(ii) The MFULIB Problem is solvable for system (14) with $\tilde{B}_0$ and λ defined in (15).

Moreover, a controller $K \in \mathcal{C}_{mf}$ of the form

$$u_k = K(y_{[0,k-1]}) \tag{16}$$

solves the MFISS Problem for system (9) if and only if the same controller K (here "the same controller" means the mapping from the measurement output to the control input is the same) solves the MFULIB Problem for the system (14) with $\tilde{B}_0$ and λ defined in (15).

Remark 5.2: Notice that the dimension of the auxiliary system (14) is two dimension higher than the original system (9).

VI. Dynamic Programming Results

Using Theorems 5.1 and the results of ULIB problems [6, Theorems 4.17, 4.19], we can obtain dynamic programming results for the MFISS Problem. They provide a framework for measurement feedback controller design to achieve ISS property.

We use $2^{\mathbf{R}^{n+2}}$ to denote the set of all subsets of $\mathbf{R}^{n+2}$, where n is the dimension of the states in system (9). We define $\hat{G} : 2^{\mathbf{R}^{n+2}} \to \mathbf{R}$ by

$$\hat{G}(X) := \sup_{(x,\zeta,\eta)\in X} \{|x| - \alpha_1(\zeta) - \gamma(\eta)\}, \quad \forall X \subseteq \mathbf{R}^{n+2} \tag{17}$$

and $F : 2^{\mathbf{R}^{n+2}} \times \mathbf{R}^m \times \mathbf{R}^p \to 2^{\mathbf{R}^{n+2}}$ by

$$\begin{array}{rcl} F(X, u, y) &=& \{(x, \zeta, \eta) : \exists w \in \mathbf{W}, \exists (x', \zeta', \eta') \in X, \\ && \text{such that } h(x', w) = y, f(x', u, w) = x, \\ && e^{-1}\zeta' = \zeta, \max\{\eta', |w|\} = \eta\}. \end{array} \tag{18}$$

The *set-valued observer* is defined as

$$X_{i+1} = F(X_i, u_i, y_i), \quad X_0 \subseteq \mathbf{R}^{n+2}. \tag{19}$$

Remark 6.1: The solution of set-valued observer are sets which are estimates of the states of system (14). In fact, for $X_0 \subseteq \mathbf{R}^n$, $j \geq 1$, $u_{[0,j-1]} \in \mathcal{U}_{[0,j-1]}$, $y_{[0,j-1]} \in \mathcal{Y}_{[0,j-1]}$,

$$\begin{array}{rcl} X_j &=& \{(x, \zeta, \eta) : \exists w_{[0,j-1]} \in \mathcal{W}_{[0,j-1]}, \exists (x_0, \zeta_0, \eta_0) \in X_0, \\ && \text{such that } x_j = x, \zeta_j = \zeta, \eta_j = \eta, \\ && h(x_i, w_i) = y_i, 0 \leq i \leq j-1, \\ && \text{where } x_{i+1} = f(x_i, u_i, w_i), \zeta_{i+1} = e^{-1}\zeta_i, \\ && \eta_{i+1} = \max\{\eta_i, |w_i|\}, 0 \leq i \leq j-1\}. \end{array} \tag{20}$$

Theorem 6.2: (Necessity) Let $\mathbf{Y} = \text{range}\{h\} \subseteq \mathbf{R}^p$ and $B_0 \subseteq \mathbf{R}^n, \mathbf{W} \subseteq \mathbf{R}^s, \mathbf{U} \subseteq \mathbf{R}^m$ be given and define the set of admissible controller $\mathcal{C}_{mf}$ as in (10). Let $\gamma \in \mathcal{K}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ be given. Let $\tilde{B}_0$ and λ come from (15). Let $\hat{G}$ come from (17) and F come from (18). If the MFISS Problem is solvable for system (9), then the value function $W_a : 2^{\mathbf{R}^{n+2}} \to \tilde{\mathbf{R}}$ defined by [1]

$$\begin{array}{rcl} W_a(X) &:=& \inf_{K\in\mathcal{C}_{mf}} \sup_{k\geq 0} \sup_{y_{[0,k-1]}\in\mathcal{Y}_{[0,k-1]}} \\ && \left\{\hat{G}(X_k) : \ X_0 = X, u_k = K(y_{[0,k-1]})\right\} \end{array} \tag{21}$$

satisfies

1) $\tilde{B}_0 \in \text{dom}W_a$ where
$$\text{dom}W_a := \left\{X \in 2^{\mathbf{R}^{n+2}} : -\infty < W_a(X) < +\infty\right\};$$
2) $W_a(\tilde{B}_0) \leq \lambda$;
3) the following dynamic programming equation (DPE) holds
$$W_a(X) = \max\{\hat{G}(X), \inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{Y}} W_a(F(X, u, y))\}, \quad \forall X \in \text{dom}W_a. \tag{22}$$

[1]Here $\mathcal{Y}_{[0,k-1]}$ is defined similarly as in (3), X_k is the solution of (19) with $u_k = K(y_{[0,k-1]})$ and $X_0 = X$.

Theorem 6.3: (Sufficiency) Let $\mathbf{Y} = \text{range}\{h\} \subseteq \mathbf{R}^p$ and $B_0 \subseteq \mathbf{R}^n, \mathbf{W} \subseteq \mathbf{R}^s, \mathbf{U} \subseteq \mathbf{R}^m$ be given and define the set of admissible controller $\mathcal{C}_{mf}$ as in (10). Let $\gamma \in \mathcal{K}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ be given. Let $\tilde{B}_0$ and λ come from (15). Let $\hat{G}$ come from (17) and F come from (18). Suppose there exist $S \subseteq 2^{\mathbf{R}^{n+2}}, W : 2^{\mathbf{R}^{n+2}} \to \tilde{\mathbf{R}}$, $\mathbf{u} : S \to \mathbf{U}$, and $X_0 \in S$ such that the following conditions hold:

1) $\tilde{B}_0 \subseteq X_0$;
2) $W(X_0) \leq \lambda$;
3) the following dynamic programming inequality (DPI) holds

$$W(X) \geq \max\{\hat{G}(X), \inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{Y}} W(F(X,u,y))\}, \quad \forall X \in S; \tag{23}$$

4) for all $X \in S$,

$$\begin{aligned} &\max\{\hat{G}(X), \sup_{y\in\mathbf{Y}} W(F(X,\mathbf{u}(X),y))\} \\ &= \max\{\hat{G}(X), \inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{Y}} W(F(X,u,y))\}; \end{aligned} \tag{24}$$

5) the solution of

$$X_{k+1} = F(X_k, \mathbf{u}(X_k), y_k) \tag{25}$$

satisfies

$$X_k \in S \tag{26}$$

for all $X_0 \in S, k \geq 0$ and $y_{[0,k-1]} \in \mathcal{Y}_{[0,k-1]}$.

Then the controller defined by

$$u_k = \mathbf{u}(X_k) \tag{27}$$

solves the MFISS Problem for system (9).

VII. Example

Consider one-dimensional discrete-time system with linear dynamics:

$$\begin{cases} x_{k+1} &= x_k + u_k + w_k, \ k \geq 0 \\ y_k &= x_k + w_k, \ k \geq 0 \end{cases} \tag{28}$$

where $x_k, u_k, y_k \in \mathbf{R}, w_k \in \mathbf{W} = \mathbf{R}$, then $\mathbf{Y} = \mathbf{R}$.

Suppose $B_0 = \mathbf{R}$, consider MFISS Problem with

$$\alpha_1(s) = s, \alpha_2(s) = es, \gamma(s) = s, \tag{29}$$

(i.e. $\beta(s,k) = se^{1-k}$).

Notice that the admissible measurement feedback controller we will choose has the form

$$u_k = K(y_{[0,k-1]}). \tag{30}$$

i.e. u_k depends on $y_0, \cdots, y_{k-1}$ but not on y_k.

We choose $X_0 = \tilde{B}_0$, since $\alpha(|x_0|) = e|x_0|$, by (15) we have

$$X_0 = \tilde{B}_0 = \{(x_0, e|x_0|, 0) : x_0 \in \mathbf{R}\}. \tag{31}$$

By (17), we have

$$\hat{G}(X_0) = \sup_{x_0\in\mathbf{R}} \{|x_0| - \zeta_0 - \eta_0\} = \sup_{x_0\in\mathbf{R}} \{|x_0| - e|x_0|\} = 0.$$

By (19) and (18), for any u_0, y_0,

$$\begin{aligned} X_1 = \{(x_1,\zeta_1,\eta_1) : \exists w_0, \exists (x_0,\zeta_0,\eta_0) \in X_0, &\text{such that} \\ x_0 + w_0 = y_0, x_0 + u_0 + w_0 = x_1&, \\ e^{-1}\zeta_0 = \zeta_1, \max\{\eta_0, |w_0|\} = \eta_1&\}. \end{aligned}$$

By $x_0 + w_0 = y_0, x_0 + u_0 + w_0 = x_1$, we have

$$\begin{aligned} w_0 &= y_0 - x_0, \\ x_1 &= y_0 + u_0. \end{aligned}$$

Hence

$$\begin{aligned} x_1 &= y_0 + u_0, \\ \eta_1 &= \max\{\eta_0, |w_0|\} = \max\{\eta_0, |y_0 - x_0|\}, \\ \zeta_1 &= e^{-1}\zeta_0 = |x_0| \end{aligned}$$

and

$$X_1 = \{(y_0 + u_0, |x_0|, |y_0 - x_0|) : x_0 \in \mathbf{R}\}.$$

By (17),

$$\hat{G}(X_1) = \sup_{x_0\in\mathbf{R}} \{|y_0 + u_0| - |x_0| - |y_0 - x_0|\}.$$

Notice that

$$\hat{G}(X_1) \geq |y_0 + u_0| - |y_0|.$$

(choose $x_0 = 0 \in \mathbf{R}$)

It is easy to see that

$$\sup_{y_0\in\mathbf{R}} \hat{G}(X_1) \leq 0$$

holds if and only if

$$u_0 = 0.$$

Now for $k \geq 1$, for any u_k, y_k, by (19) and (18) we have

$$\begin{aligned} X_{k+1} \;=\; &\{(y_k + u_k, e^{-k}|x_0|, \max\{|y_0 - x_0|, \\ &|y_1 - y_0 - u_0|, \cdots, |y_k - y_{k-1} - u_{k-1}|\}) : \\ &x_0 \in \mathbf{R}\}. \end{aligned}$$

Again by (17),

$$\begin{aligned} \hat{G}(X_{k+1}) \;=\; &\sup_{x_0\in\mathbf{R}} \{|y_k + u_k| - e^{-k}|x_0| \\ &- \max\{|y_k - y_{k-1} - u_{k-1}|, \cdots, \\ &|y_1 - y_0 - u_0|, |y_0 - x_0|\}\} \\ \geq\; &|y_k + u_k| - \max\{|y_k - y_{k-1} - u_{k-1}|, \\ &\cdots, |y_1 - y_0 - u_0|, |y_0|\}. \end{aligned}$$

For $|y_k|$ sufficiently large,

$$\begin{aligned} |y_k - y_{k-1} - u_{k-1}| \geq \max\{&|y_{k-1} - y_{k-2} - u_{k-2}|, \\ &\cdots, |y_1 - y_0 - u_0|, |y_0|\}, \end{aligned}$$

and

$$\hat{G}(X_{k+1}) \geq |y_k + u_k| - |y_k - y_{k-1} - u_{k-1}|.$$

It is easy to see that

$$\sup_{y_k\in\mathbf{R}} \hat{G}(X_{k+1}) \leq 0,$$

holds if and only if

$$u_k = -y_{k-1} - u_{k-1}.$$

So we obtain a control law

$$u_0 = 0, \quad u_k = -y_{k-1} - u_{k-1} = \sum_{i=0}^{k-1} (-1)^{k+i} y_i, \; k \geq 1. \tag{32}$$

We will prove that (32) is the optimal controller. Now we use S to denote the set of all the possible set-valued observer obtained by (19) when the controller is given by (32) (output $y_k (k \geq 0)$ are arbitrary). i.e.

$$S = \{X_j : j \geq 0, y_k \in \mathbf{R}(k \geq 0), u_0 = 0, \\ u_k = \textstyle\sum_{i=0}^{k-1} (-1)^{k+i} y_i, k \geq 1.\}$$

where X_0 is given by (31).

By induction, we can prove the value function $W_a(X)$ defined by (21) satisfies

$$W_a(X) = 0, \quad \forall X \in S.$$

Since

$$\hat{G}(X) \leq 0, \quad \forall X \in S,$$

it is easy to see that $W_a(X)$ satisfies the dynamic programming equation

$$W_a(X) = \max\{\hat{G}(X), \inf_{u \in \mathbf{R}} \sup_{y \in \mathbf{R}} W_a(F(X, u, y))\}, \quad \forall X \in S \tag{33}$$

where

$$F(X, u, y) = \{(y + u, e^{-1}\zeta, \max\{\eta, |y - x|\}) : (x, \zeta, \eta) \in X\}.$$

From Theorem 6.3, the controller (32) is the optimal controller such that the closed-loop system is ISS with γ and β. In fact, using the controller (32), the closed-loop system becomes

$$\left\{ \begin{array}{rcl} x_0 & = & x_0, \\ x_1 & = & x_0 + w_0, \\ x_{k+1} & = & w_k, \; k \geq 1 \end{array} \right. \tag{34}$$

Obviously it is ISS with

$$\begin{array}{rcl} \gamma_a(s) & = & s = \gamma(s), \\ \beta_a(s,k) & = & \left\{ \begin{array}{ll} s, & k = 0 \\ s, & k = 1 \\ 0, & k \geq 2 \end{array} \right. \leq \beta(s,k) = se^{1-k}. \end{array}$$

Remark 7.1: The above example is a very special one dimensional example. Only for very special examples, it is possible to obtain an explicit solution. In general, the set-valued observer is not easy to obtain, this makes the solving of the dynamic programming equation (inequality) very difficult.

VIII. CONCLUSION AND FUTURE WORK

In this paper, we considered the controller synthesis to achieve the ISS property. We made a connection between the ISS property and the l^∞ bounded robustness considered in [6]. It turns out that the design methods provided in [6] is a powerful tool that can be applied to the synthesis of ISS property when the disturbances gain and the transient bound are prescribed. The measurement feedback ISS synthesis problem can be solved in principle using dynamic programming techniques. Though we only considered ISS synthesis problem in this paper, our method can be easily used for many other synthesis problems for ISS-like properties. Further research include the synthesis problems to achieve the optimal/suboptimal gains, and the reduction of the computation complexity.

REFERENCES

[1] D. Angeli, *A Lyapunov approach to incremental stability properties*, IEEE Trans. Aut. Contr., **47**, 410-422, 2002.

[2] D. Angeli, E.D. Sontag and Y. Wang, *A characterization of integral input-to-state stability*, IEEE Trans. Aut. Control, **45**, 1082-1097, 2000.

[3] T. Basar and P. Bernhard, "H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach", 2nd Ed., Birkhauser, Boston, 1995.

[4] M.A. Dahleh and I.J. Diaz-Bobillo, "Control of Uncertain Systems: A Linear Programming Approach", Prentice-Hall, Englewood Cliffs, NJ, 1995.

[5] J.W. Helton and M.R. James, "Extending H^∞ Control to Nonlinear Systems: Control of Systems to Achieve Performance Objectives", Advances in Design and Control, SIAM, 1999.

[6] S. Huang and M.R. James, *l^∞-Bounded Robustness for Nonlinear Systems: Analysis and Synthesis*, IEEE Trans. Aut. Control, **48**(11), 1875-1891, 2003.

[7] S. Huang, M.R. James, D. Nešić and P. Dower, *Analysis of Input to State Stability for Discrete Time Nonlinear Systems via Dynamic Programming*, submitted to Automatica, 2002.

[8] S. Huang, M.R. James, D. Nešić and P. Dower, *On input to state stabilization for nonlinear discrete-time systems: a dynamic programming approach*. Proc. of the 2003 European Control Conference, University of Cambridge, UK, September 1-4, 2003.

[9] S. Huang, M.R. James, D. Nešić and P. Dower, *A unified approach to controller design for achieving ISS and related properties*. Submitted to IEEE Trans. Aut. Control.

[10] Z.P. Jiang, A.R. Teel and L.Praly, *Small-gain theorem for ISS systems and applications*, Mathematics of Control, Signals, and Systems, **7**, 95-120, 1994.

[11] Z.P. Jiang and Y. Wang, *Input-to-state stability for discrete-time nonlinear systems*, Automatica, **37**, 857-869, 2001.

[12] M. Krichman, E.D. Sontag and Y. Wang, *Input-output-to-state stability*, SIAM J. Control and Optimization, **39**(6), 1874-1928, 2001.

[13] D. Liberzon, E.D. Sontag and Y. Wang, *Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation*, Systems & Control Letters, **46**, 111-127, 2002.

[14] E.D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Aut. Control, **34**, 435-443, 1989.

[15] E.D. Sontag, *Comments on integral variants of ISS*, Systems & Control Letters, **34**, 93-100, 1998.

[16] E.D. Sontag, *The ISS philosophy as a unifying framework for stability-like behavior*, Nonlinear Control in the Year 2000 (Volume 2) (Lecture Notes in Control and Information Sciences, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds.), Springer-Verlag, Berlin, 2000, pp. 443-468.

[17] J.S. Shamma and K.-Y. Tu, *Set-Valued Observers and Optimal Disturbance Rejection*, IEEE Trans. Aut. Control, **42**(2), 253-264, 1999.

[18] E.D. Sontag and Y. Wang, *On characterizations of input to state stability property*, Systems & Control Letters, **24**, 351-359, 1995.

[19] E.D. Sontag and Y. Wang, *New characterizations of input to state stability property*, IEEE Trans. Aut. Control, **41**, 1283-1294, 1996.

[20] E.D. Sontag and Y. Wang, *Notions of input to output stability*, Systems & Control Letters, **38**, 235-248, 1999.

[21] A.J. van der Schaft, "L_2-Gain and Passivity Techniques in Nonlinear Control", Springer Verlag, 1996.