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Reliable linear-quadratic control for symmetric composite systems

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The reliable linear–quadratic (LQ) state feedback control for symmetric composite systems is studied. Necessary and sufficient conditions for a state feedback controller to be a reliable LQ controller are given by using the special structure of symmetric composite systems. Simple controller design methods are presented and an illustrative example is given.

1. Introduction

In recent years there has been great interest in the study of symmetric composite systems. Symmetric composite systems are systems composed of identical subsystems which are symmetrically interconnected. The motivation for studying this class of systems is due to its very diverse application areas, such as in electric power systems, industrial manipulators and computer networks (Lunze 1986, Sundareshan and Elbanna 1991, Hovd and Skogestad 1994). It is shown that many analysis and control problems for symmetric composite systems can be simplified because of their special structure. Lunze (1986) first proposed the state-space model of symmetric composite systems and studied some fundamental properties of the systems. Sundareshan and Elbanna (1991) considered the decentralized control of the systems. Liu (1992) studied the output regulation for symmetric composite systems. Lam and Yang (1996) investigated the model reduction problem of such systems. H_2-, H_∞- and μ-optimal control problems were considered by Hovd and Skogestad (1994) and Hovd *et al.* (1997). The centralized and decentralized control for uncertain symmetric composite systems were studied by Yang and Zhang (1995) and Bakule and Rdellar (1996) respectively.

Sometimes, control systems may result in unsatisfactory performance or even instability in the event of control component failures. Recently, Veillette *et al.* (1992) and Veillette (1995) considered the design of reliable control systems. The resulting control systems provide guaranteed stability and satisfy an H_∞-norm disturbance attenuation bound or given performance bound not only when all control components are operational, but also in the case of actuator or sensor outages in the systems. The outages were restricted to occur within a pre-selected subset of available measurement or control inputs. Reliable control using redundant controllers was studied by Yang *et al.* (1998). For the reliable control of symmetric composite systems, Yang *et al.* (1996) studied the primary contingency case of a reliable H_∞ controller design problem; Huang *et al.* (1999) considered the decentralized H_∞ control problem and the fault tolerance of the designed systems.

This paper is concerned with the reliable linear–quadratic (LQ) controller design for symmetric composite systems. Both centralized and decentralized reliable controller design problems are studied. Several cases of actuator outages are considered. By using the special structure of symmetric composite systems and the results of Veillette (1995), necessary and sufficient conditions for a state feedback controller to be a reliable LQ controller are given and some simple controller design methods are presented. The rest of this paper is organized as follows. In § 2 the state-space description of the system concerned is given and the problem statement is presented. Some notation and lemmas are given in § 3. Necessary and sufficient conditions for several kinds of reliable controller to exist are given and some reliable controller design methods are presented in § 4. In § 5, an example is given to illustrate the methodology. Finally, § 6 concludes the paper.

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2. System description and problem formulation

In this paper, for given matrices $Y, Z \in \mathbb{R}^{p\times q}$ and positive integers k, j, we define

$$\bar{S}(Y,Z,k) = \begin{bmatrix} Y & Z & \dots & Z \\ Z & Y & \dots & Z \\ \vdots & \vdots & \ddots & \vdots \\ Z & Z & \dots & Y \end{bmatrix} \in \mathbb{R}^{kp\times kq},$$

$$(\bar{S}(Y,Z,1) = Y),$$

$$\tilde{S}(Y,Z,k,j) = \begin{bmatrix} Y & Z & \dots & Z \\ Z & Z & \dots & Z \\ \vdots & \vdots & \dots & \vdots \\ Z & Z & \dots & Z \end{bmatrix} \in \mathbb{R}^{kp\times jq}.$$

The symmetric composite system under consideration consists of N subsystems, the ith subsystem is described by

$$\dot{x}_i = A_1 x_i + \sum_{k=1,k\neq i}^{N} A_{12} x_k + B_1 u_i,$$

where $i = 1,2,\dots,N$ and $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ $(i = 1,\dots,N)$ are the n,m-dimensional state and control input respectively; A_1, $A_{12} \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$. Then the overall system is given by

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x = (x_1^{\mathrm{T}},\dots,x_N^{\mathrm{T}})^{\mathrm{T}}$, $u = (u_1^{\mathrm{T}},\dots,u_N^{\mathrm{T}})^{\mathrm{T}}$ and

$$\begin{aligned} A &= \bar{S}(A_1, A_{12}, N) \in \mathbb{R}^{Nn\times Nn}, \\ B &= \operatorname{diag}[B_1,\dots,B_1] \in \mathbb{R}^{Nn\times Nm}. \end{aligned} \tag{2}$$

In this paper, we shall consider the reliable LQ regulator problem (Veillette 1995) for system (1).

Suppose that the quadratic performance index is given by

$$J = \int_0^\infty (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u)\,\mathrm{d}t,$$

where

$$Q = \operatorname{diag}[Q_1,\dots,Q_1] \in \mathbb{R}^{Nn\times Nn},$$

$$R = \operatorname{diag}[R_1,\dots,R_1] \in \mathbb{R}^{Nm\times Nm},$$

$Q_1 \in \mathbb{R}^{n\times n} \geqslant 0$ is a constant matrix and $R_1 \in \mathbb{R}^{m\times m} > 0$ is a diagonal constant matrix. The problem is to design a reliable LQ state-feedback regulator. The design should tolerate outages within a subset of actuators, while maintaining stability and a known quadratic performance bound. Considering the special structure of symmetric composite systems, we suppose that a particular subset of actuators within each subsystem is prone to failure, and there may be an upper limit to the number of subsystems with faulty actuators at the same time. Both centralized and decentralized reliable controller design problems are considered.

3. Notation and lemmas

The following notation and lemmas are needed in this paper.

For a positive integer p, define

$$m_k = [1 \quad v_k \quad v_k^2 \quad \cdots \quad v_k^{p-1}]^{\mathrm{T}}, \quad k = 1,2,\dots,p,$$

where $v_k = \exp[2\pi(k-1)\mathrm{j}/p]$, $k = 1,2,\dots,p$, $(\mathrm{j} = (-1)^{1/2})$, that is v_k is a root of the equation $v^p = 1$.

Let $t = (p+1)/2$ if p is odd, and $t = p/2$ if p is even. Define $r_1 = m_1 = [1\ 1\ \cdots\ 1]^{\mathrm{T}}$, $r_{p/2+1} = m_{p/2+1}$ if p is an even number, $r_i = (1/2^{1/2})(m_i + m_{p+2-i})$, $r_{p+2-i} = (\mathrm{j}/2^{1/2})(m_i - m_{p+2-i})$ $(i = 2,3,\dots,t)$. Define

$$R_p = \frac{1}{p^{1/2}}[r_1 \quad r_2 \quad \cdots \quad r_p]. \tag{3}$$

From the results of Hovd and Skogestad (1994) and Huang *et al.* (1999), R_p is a real orthogonal matrix, and the following two lemmas hold.

Lemma 1 (Hovd and Skogestad 1994): *For an integer $p \geqslant 2$, we have*

$$R_p^{-1}\bar{S}(a,b,p)R_p = \operatorname{diag}[a+(p-1)b, a-b,\dots,a-b] \in \mathbb{R}^{p\times p},$$

where a and b are two arbitrarily given numbers.

Lemma 2 (Huang *et al.* 1999): *For integers $p \geqslant 1$ and $q \geqslant 1$, we have*

$$R_p^{-1}\tilde{S}(1,1,p,q)R_q = \tilde{S}((pq)^{1/2},0,p,q) \in \mathbb{R}^{p\times q}.$$

In this paper, we further define

$$T_{pi} = R_p \otimes I_i, \tag{4}$$

where R_p is given in (3), $I_i \in \mathbb{R}^{i\times i}$ is the identity matrix and $\otimes$ denotes the Kronecker product.

From lemmas 1 and 2, we can prove the following two lemmas.

Lemma 3: *Suppose that $l \leqslant N-1$ is a positive integer,*

$$M = \begin{bmatrix} \bar{S}(M_1, M_{11}, l) & \tilde{S}(M_{12}, M_{12}, l, N-l) \\ \tilde{S}(M_{21}, M_{21}, N-l, l) & \bar{S}(M_2, M_{22}, N-l) \end{bmatrix} \in \mathbb{R}^{Nn\times Nn}, \tag{5}$$

where $M_1, M_{11}, M_2, M_{22}, M_{12}, M_{21} \in \mathbb{R}^{n\times n}$. Then there exists an orthogonal matrix P such that

$$P^{\mathrm{T}}MP = P^{-1}MP = \operatorname{diag}\left[\bar{M}_{l0}, \overbrace{\bar{M}_1, \ldots, \bar{M}_1}^{l-1}, \overbrace{\bar{M}_2, \ldots, \bar{M}_2}^{N-l-1}\right] \tag{6}$$

where

$$\bar{M}_{l0} = \begin{bmatrix} M_1 + (l-1)M_{11} & [l(N-l)]^{1/2}M_{12} \\ [l(N-l)]^{1/2}M_{21} & M_2 + (N-l-1)M_{22} \end{bmatrix},$$

$$\bar{M}_1 = M_1 - M_{11}, \quad \bar{M}_2 = M_2 - M_{22}.$$

Proof: From lemmas 1 and 2 and from (4), we have

$$T_{ln}^{-1}\bar{S}(M_1, M_{11}, l)T_{ln} = \operatorname{diag}\left[M_1 + (l-1)M_{11}, \overbrace{\bar{M}_1, \ldots, \bar{M}_1}^{l-1}\right],$$

$$T_{(N-l)n}^{-1}\bar{S}(M_2, M_{22}, N-l)T_{(N-l)n} = \operatorname{diag}\left[M_2 + (N-l-1)M_{22}, \overbrace{\bar{M}_2, \ldots, \bar{M}_2}^{N-l-1}\right]$$

and

$$T_{ln}^{-1}\tilde{S}(M_{12}, M_{12}, l, N-l)T_{(N-l)n} = \tilde{S}([l(N-l)]^{1/2}M_{12}, 0, l, N-l),$$

$$T_{(N-l)n}^{-1}\tilde{S}(M_{21}, M_{21}, N-l, l)T_{ln} = \tilde{S}([l(N-l)]^{1/2}M_{21}, 0, N-l, l).$$

Thus

$$\begin{bmatrix} T_{ln}^{-1} & 0 \\ 0 & T_{(N-l)n}^{-1} \end{bmatrix} M \begin{bmatrix} T_{ln} & 0 \\ 0 & T_{(N-l)n} \end{bmatrix} = \begin{bmatrix} \operatorname{diag}\left[M_1 + (l-1)M_{11}, \overbrace{\bar{M}_1, \ldots, \bar{M}_1}^{l-1}\right] & \tilde{S}([l(N-l)]^{1/2}M_{12}, 0, l, N-l) \\ \tilde{S}([l(N-l)]^{1/2}M_{21}, 0, N-l, l) & \operatorname{diag}\left[M_2 + (N-l-1)M_{22}, \overbrace{\bar{M}_2, \ldots, \bar{M}_2}^{N-l-1}\right] \end{bmatrix}.$$

Since

$$\begin{bmatrix} T_{ln} & 0 \\ 0 & T_{(N-l)n} \end{bmatrix}$$

is an orthogonal matrix, there exists an orthogonal matrix P such that (6) holds. □

Lemma 4: *Suppose that $l \leqslant N-1$ is a positive integer, $M \in \mathbb{R}^{Nn\times Nn}$ has the structure (5) and symmetric matrices E and F have the structures*

$$E = \operatorname{diag}\left[\overbrace{E_1, \ldots, E_1}^{l}, \overbrace{E_2, \ldots, E_2}^{N-l}\right] \in \mathbb{R}^{Nn\times Nn}, \tag{7}$$

$$F = \begin{bmatrix} \bar{S}(F_1, F_{11}, l) & \tilde{S}(F_{12}, F_{12}, l, N-l) \\ \tilde{S}(F_{12}^{\mathrm{T}}, F_{12}^{\mathrm{T}}, N-l, l) & \bar{S}(F_2, F_{22}, N-l) \end{bmatrix} \in \mathbb{R}^{Nn\times Nn}, \tag{8}$$

where $E_1, E_2, F_1, F_{11}, F_2, F_{22}, F_{12} \in \mathbb{R}^{n\times n}$. If the algebraic Riccati equation

$$M^{\mathrm{T}}X + XM - XEX + F = 0 \tag{9}$$

has a unique solution X, then X has the same structure as M, that is

$$X = \begin{bmatrix} \bar{S}(X_1, X_{11}, l) & \tilde{S}(X_{12}, X_{12}, l, N-l) \\ \tilde{S}(X_{21}, X_{21}, N-l, l) & \bar{S}(X_2, X_{22}, N-l) \end{bmatrix} \in \mathbb{R}^{Nn\times Nn}, \tag{10}$$

where $X_1, X_2, X_{11}, X_{22}, X_{12}, X_{21} \in \mathbb{R}^{n\times n}$. Moreover, X can be calculated by

$$X_1 = \frac{\bar{X}_{l0}^{11} + (l-1)\bar{X}_1}{l}, \quad X_{12} = \frac{\bar{X}_{l0}^{12}}{[l(N-l)]^{1/2}},$$
$$X_{21} = X_{12}^{\mathrm{T}}, \tag{11}$$

$$X_{11} = \frac{\bar{X}_{l0}^{11} - \bar{X}_1}{l}, \quad X_2 = \frac{\bar{X}_{l0}^{22} + (N-l-1)\bar{X}_2}{N-l},$$
$$X_{22} = \frac{\bar{X}_{l0}^{22} - \bar{X}_2}{N-l}, \tag{12}$$

where $\bar{X}_{l0}, \bar{X}_1$ and $\bar{X}_2$ are the unique solutions of the algebraic Riccati equations

$$\bar{M}_{l0}^{\mathrm{T}}\bar{X}_{l0} + \bar{X}_{l0}\bar{M}_{l0} - \bar{X}_{l0}\operatorname{diag}[E_1, E_2]\bar{X}_{l0} + \bar{F}_{l0} = 0, \tag{13}$$

$$\bar{M}_1^{\mathrm{T}}\bar{X}_1 + \bar{X}_1\bar{M}_1 - \bar{X}_1E_1\bar{X}_1 + \bar{F}_1 = 0, \tag{14}$$

$$\bar{M}_2^{\mathrm{T}}\bar{X}_2 + \bar{X}_2\bar{M}_2 - \bar{X}_2E_2\bar{X}_2 + \bar{F}_2 = 0, \tag{15}$$

respectively, and

$$\bar{X}_{l0} = \begin{bmatrix} \bar{X}_{l0}^{11} & \bar{X}_{l0}^{12} \\ (\bar{X}_{l0}^{12})^{\mathrm{T}} & \bar{X}_{l0}^{22} \end{bmatrix}, \tag{16}$$

$$\bar{F}_{l0} = \begin{bmatrix} F_1 + (l-1)F_{11} & [l(N-l)]^{1/2}F_{12} \\ [l(N-l)]^{1/2}F_{12}^{\mathrm{T}} & F_2 + (N-l-1)F_{22} \end{bmatrix},$$

$$\bar{F}_1 = F_1 - F_{11}, \quad \bar{F}_2 = F_2 - F_{22}.$$

Proof: Suppose that

$$X = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} & \dots & \hat{X}_{1l} & \hat{X}_{1(l+1)} & \hat{X}_{1(l+2)} & \dots & \hat{X}_{1N} \\ \hat{X}_{21} & \hat{X}_{22} & \dots & \hat{X}_{2l} & \hat{X}_{2(l+1)} & \hat{X}_{2(l+2)} & \dots & \hat{X}_{2N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ \hat{X}_{l1} & \hat{X}_{l2} & \dots & \hat{X}_{ll} & \hat{X}_{l(l+1)} & \hat{X}_{l(l+2)} & \dots & \hat{X}_{lN} \\ \hat{X}_{(l+1)1} & \hat{X}_{(l+1)2} & \dots & \hat{X}_{(l+1)l} & \hat{X}_{(l+1)(l+1)} & \hat{X}_{(l+1)(l+2)} & \dots & \hat{X}_{(l+1)N} \\ \hat{X}_{(l+2)1} & \hat{X}_{(l+2)2} & \dots & \hat{X}_{(l+2)l} & \hat{X}_{(l+2)(l+1)} & \hat{X}_{(l+2)(l+2)} & \dots & \hat{X}_{(l+2)N} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{X}_{N1} & \hat{X}_{N2} & \dots & \hat{X}_{Nl} & \hat{X}_{N(l+1)} & \hat{X}_{N(l+2)} & \dots & \hat{X}_{NN} \end{bmatrix},$$

where $\hat{X}_{ij} \in \mathbb{R}^{n\times n}(1 \leqslant i,j \leqslant N)$. In order to prove that X has the structure of (10), it is sufficient to show that

$$\begin{aligned} &\hat{X}_{ii} = \hat{X}_{11}, \quad \hat{X}_{i'i'} = \hat{X}_{NN}, \quad \hat{X}_{ij} = \hat{X}_{12}, \quad \hat{X}_{ij'} = \hat{X}_{1N}, \\ &\hat{X}_{i'j} = \hat{X}_{N1}, \quad \hat{X}_{i'j'} = \hat{X}_{N(l+1)}, \\ &\qquad \forall 1 \leqslant i, \quad j \leqslant l, \quad l+1 \leqslant i', \quad j' \leqslant N, \\ &\qquad i \neq j, \quad i' \neq j'. \end{aligned} \tag{17}$$

For $2 \leqslant i \leqslant l$, define

$$H_i = \left[\begin{array}{c|c} \overbrace{\begin{matrix} 0 & 0 & \dots & 0 & I_n \\ 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \\ I_n & 0 & \dots & 0 & 0 \end{matrix}}^{i} & 0 \\ \hline 0 & \underbrace{\begin{matrix} I_n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_n \end{matrix}}_{N-i} \end{array}\right].$$

Then $H_i^{\mathrm{T}} = H_i^{-1} = H_i$, $H_iMH_i = M$, $H_iEH_i = E$, and $H_iFH_i = F$. Hence

$$\begin{aligned} 0 &= M^{\mathrm{T}}X + XM - XEX + F \\ &= H_i(M^{\mathrm{T}}X + XM - XEX + F)H_i \\ &= H_iM^{\mathrm{T}}H_iH_iXH_i + H_iXH_iH_iMH_i \\ &\quad - H_iXH_iH_iEH_iH_iXH_i + H_iFH_i \\ &= M^{\mathrm{T}}(H_iXH_i) + (H_iXH_i)M \\ &\quad - (H_iXH_i)E(H_iXH_i) + F. \end{aligned}$$

The above equation shows that H_iXH_i is also a solution of (9). Since the solution of (9) is unique, we have

$$X = H_iXH_i = \begin{bmatrix} \hat{X}_{ii} & \hat{X}_{i2} & \dots & \hat{X}_{i1} & \dots & \hat{X}_{il} & \dots & \hat{X}_{iN} \\ \hat{X}_{2i} & \hat{X}_{22} & \dots & \hat{X}_{21} & \dots & \hat{X}_{2l} & \dots & \hat{X}_{2N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ \hat{X}_{1i} & \hat{X}_{12} & \dots & \hat{X}_{11} & \dots & \hat{X}_{1l} & \dots & \hat{X}_{1N} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \dots & \vdots \\ \hat{X}_{li} & \hat{X}_{l2} & \dots & \hat{X}_{l1} & \dots & \hat{X}_{ll} & \dots & \hat{X}_{lN} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{X}_{Ni} & \hat{X}_{N2} & \dots & \hat{X}_{N1} & \dots & \hat{X}_{Nl} & \dots & \hat{X}_{NN} \end{bmatrix}.$$

This implies that

$$\begin{aligned} &\hat{X}_{ii} = \hat{X}_{11}, \quad \hat{X}_{i1} = \hat{X}_{1i}, \quad \hat{X}_{ij} = \hat{X}_{1j}, \quad \hat{X}_{ji} = \hat{X}_{j1}, \\ &\qquad \forall 2 \leqslant j \leqslant N, \quad j \neq i \quad (2 \leqslant i \leqslant l) \end{aligned} \tag{18}$$

Similarly, for $l+1 \leqslant i' \leqslant N-1$, define

$$G_{i'} = \left[\begin{array}{c|c} \underbrace{\begin{matrix} I_n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_n \end{matrix}}_{i'-1} & 0 \\ \hline 0 & \overbrace{\begin{matrix} 0 & 0 & \dots & 0 & I_n \\ 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \\ I_n & 0 & \dots & 0 & 0 \end{matrix}}^{N-i'+1} \end{array}\right].$$

Then we can obtain

$$0 = M^{\mathrm{T}}(G_{i'}XG_{i'}) + (G_{i'}XG_{i'})M - (G_{i'}XG_{i'})E(G_{i'}XG_{i'}) + F$$

and $X = G_{i'}XG_{i'}$, hence

$$\hat{X}_{i'i'} = \hat{X}_{NN}, \quad \hat{X}_{i'N} = \hat{X}_{Ni'}, \quad \hat{X}_i = \hat{X}_{Nj}, \quad \hat{X}_{ji'} = \hat{X}_{jN}, \quad \forall\, 1 \leqslant j \leqslant N-1, \quad j \neq i' \quad (l+1 \leqslant i' \leqslant N-1). \tag{19}$$

From (18) and (19), (17) is obtained. Hence X has the structure of (10).

Moreover, define

$$\bar{X}_{l0} = \begin{bmatrix} X_1 + (l-1)X_{11} & [l(N-l)]^{1/2}X_{12} \\ [l(N-l)]^{1/2}X_{21} & X_2 + (N-l-1)X_{22} \end{bmatrix}, \tag{20}$$

$$\bar{X}_1 = X_1 - X_{11}, \quad \bar{X}_2 = X_2 - X_{22}. \tag{21}$$

From lemma 3, we have

$$\begin{aligned} P^{\mathrm{T}}XP &= P^{-1}XP \\ &= \operatorname{diag}\Big[\bar{X}_{l0}, \overbrace{\bar{X}_1, \ldots, \bar{X}_1}^{l-1}, \overbrace{\bar{X}_2, \ldots, \bar{X}_2}^{N-l-1}\Big], \\ P^{\mathrm{T}}EP &= P^{-1}EP \\ &= \operatorname{diag}\Big[E_1, E_2, \overbrace{E_1, \ldots, E_1}^{l-1}, \overbrace{E_2, \ldots, E_2}^{N-l-1}\Big], \\ P^{\mathrm{T}}FP &= P^{-1}FP \\ &= \operatorname{diag}\Big[\bar{F}_{l0}, \overbrace{\bar{F}_1, \ldots, \bar{F}_1}^{l-1}, \overbrace{\bar{F}_2, \ldots, \bar{F}_2}^{N-l-1}\Big]. \end{aligned} \tag{22}$$

Since the Riccati equation (9) is equivalent to

$$P^{\mathrm{T}}(M^{\mathrm{T}}X + XM - XEX + F)P = 0,$$

that is,

$$(P^{\mathrm{T}}MP)^{\mathrm{T}}(P^{\mathrm{T}}XP) + (P^{\mathrm{T}}XP)(P^{\mathrm{T}}MP) - (P^{\mathrm{T}}XP)(P^{\mathrm{T}}EP)(P^{\mathrm{T}}XP) + P^{\mathrm{T}}FP = 0,$$

from (6) and (22), it is equivalent to the three Riccati equations (13)–(15). Equations (11) and (12) can be easily obtained from (16), (20) and (21). The proof is completed. □

Remark: Lemma 4 shows that for the matrices M, E and F with the structures (5), (7) and (8) respectively, the solution X of the corresponding algebraic Riccati equation has the same structure as M. This result is obviously true for the case of Lyapunov equation. So Lemma 4 is the generalization of theorems 1 and 2 of Sundareshan and Elbanna (1991). □

4. Reliable controller design

In this section, we consider the reliable LQ controller design problem for system (1). We suppose that there is an upper limit to the number of subsystems with faulty actuators. We use l_0 to denote this number. In §4.1, the centralized reliable LQ controller design problem is discussed. In §4.2, the decentralized reliable LQ controller design problem is considered.

In the following, we define

$$A_s = A_1 - A_{12}, \quad A_t = A_1 + (N-1)A_{12},$$

$$A_{l0} = \begin{bmatrix} A_1 + (l-1)A_{12} & [l(N-l)]^{1/2}A_{12} \\ [l(N-l)]^{1/2}A_{12} & A_1 + (N-l-1)A_{12} \end{bmatrix}, \quad 1 \leqslant l \leqslant N-1.$$

4.1. *Centralized reliable controller design*

In this section, we consider the problem to design a centralized state feedback controller

$$u = \tilde{K}x \, (\tilde{K} \in \mathbb{R}^{Nm \times Nn})$$

for system (1) such that the closed-loop system maintains stability and a known quadratic performance bound when outages occur within actuators of any $l \leqslant l_0$ subsystems. Because, in system (1), all the subsystems are identical, and all the interconnections between the subsystems are also identical, it is easy to see that the gain matrix $\tilde{K}$ should have the following structure:

$$\tilde{K} = \bar{S}(\tilde{K}_1, \tilde{K}_2, N) \in \mathbb{R}^{Nm \times Nn}, \tag{23}$$

where $\tilde{K}_1, \tilde{K}_2 \in \mathbb{R}^{m \times n}$.

We first consider a special case. Suppose the fact that actuator outages occur in a subsystem means that all the actuators in the subsystem fail simultaneously. The following theorem gives a necessary and sufficient condition for a controller of the form (23) to be the required reliable controller.

Theorem 1: *Suppose that the performance bound is J_0, the initial state of the system is x_0, then the controller of the form* (23) *is a reliable controller which can tolerate l_0* $(1 \leqslant l_0 \leqslant N-1)$ *subsystems faults if and only if $\tilde{K}_1$ and $\tilde{K}_2$ satisfy the following two conditions.*

$$\text{(P1)} \quad x_0^{\mathrm{T}} T_{Nn} \operatorname{diag}\Big[W_{tK}, \overbrace{W_{sK}, \ldots, W_{sK}}^{N-1}\Big] T_{Nn}^{-1} x_0 \leqslant J_0, \tag{24}$$

where T_{Nn} is defined in (4), *and W_{sK} and W_{tK} are the unique positive definite solutions of the Lyapunov equations*

$$A_{sK}^{\mathrm{T}} W_{sK} + W_{sK} A_{sK} + Q_1 + (\tilde{K}_1 - \tilde{K}_2)^{\mathrm{T}} R_1 (\tilde{K}_1 - \tilde{K}_2) = 0 \qquad (25)$$

and

$$A_{tK}^{\mathrm{T}} W_{tK} + W_{tK} A_{tK} + Q_1 + [\tilde{K}_1 + (N-1)\tilde{K}_2]^{\mathrm{T}} \times R_1 [\tilde{K}_1 + (N-1)\tilde{K}_2] = 0 \quad (26)$$

respectively, where

$$A_{sK} = A_s + B_1(\tilde{K}_1 - \tilde{K}_2),$$
$$A_{tK} = A_t + B_1[\tilde{K}_1 + (N-1)\tilde{K}_2]$$

(P2) *For* $1 \leqslant l \leqslant l_0$,

$$x_0^{\mathrm{T}} P \,\mathrm{diag}\left[W_{l0K}, \overbrace{W_s, \ldots, W_s}^{l-1}, \overbrace{W_{sK}, \ldots, W_{sK}}^{N-l-1} \right] P^{-1} x_0 \leqslant J_0, \qquad (27)$$

where P is the orthogonal matrix in lemma 3, W_{sK} is the solution of (25), *and W_{l0K} and W_s are the unique positive definite solutions of the Lyapunov equations*

$$A_{l0K}^{\mathrm{T}} W_{l0K} + W_{l0K} A_{l0K} + \mathrm{diag}\,[Q_1, Q_1] + \begin{bmatrix} \bar{K}_{l0}^{11} & \bar{K}_{l0}^{12} \\ (\bar{K}_{l0}^{12})^{\mathrm{T}} & \bar{K}_{l0}^{22} \end{bmatrix} = 0 \qquad (28)$$

and

$$A_s^{\mathrm{T}} W_s + W_s A_s + Q_1 = 0 \qquad (29)$$

respectively, where

$$A_{l0K} = A_{l0} + \begin{bmatrix} 0 & 0 \\ [l(N-l)]^{1/2} B_1 \tilde{K}_2 & B_1[\tilde{K}_1 + (N-l-1)\tilde{K}_2] \end{bmatrix},$$

$$\bar{K}_{l0}^{11} = l(N-l)\tilde{K}_2^{\mathrm{T}} R_1 \tilde{K}_2,$$
$$\bar{K}_{l0}^{12} = [l(N-l)]^{1/2} \tilde{K}_2^{\mathrm{T}} R_1 [\tilde{K}_1 + (N-l-1)\tilde{K}_2],$$
$$\bar{K}_{l0}^{22} = [\tilde{K}_1^{\mathrm{T}} + (N-l-1)\tilde{K}_2^{\mathrm{T}}] \times R_1[\tilde{K}_1 + (N-l-1)\tilde{K}_2].$$

Proof: From lemma 1 and the special structure of A, B and $\tilde{K}$, when no actuator outage occurs, the closed-loop system is stable if and only if A_{sK} and A_{tK} are stable. This is equivalent to when the Lyapunov equations (25) and (26) have unique positive definite solutions.

When no actuator outage occurs, the performance index is

$$J = x_0^{\mathrm{T}} W x_0,$$

where

$$W = \int_0^{\infty} \exp\{(A + B\tilde{K})^{\mathrm{T}} t\}[Q + \tilde{K}^{\mathrm{T}} R \tilde{K}] \times \exp\{(A + B\tilde{K})t\}\,\mathrm{d}t$$

and W can be computed by solving the Lyapunov equation

$$(A + B\tilde{K})^{\mathrm{T}} W + W(A + B\tilde{K}) + Q + \tilde{K}^{\mathrm{T}} R \tilde{K} = 0.$$

From theorem 2 of Sundareshan and Elbanna (1991),

$$T_{Nn}^{-1} W T_{Nn} = \mathrm{diag}\left[W_{tK}, \overbrace{W_{sK}, \ldots, W_{sK}}^{N-1} \right],$$

where W_{sK} and W_{tK} are the solutions of (25) and (26) respectively. Hence $J \leqslant J_0$ if and only if (24) holds.

From lemma 3, when the actuators of l subsystems outages occur, the resultant closed-loop system is stable if and only if A_{sK}, A_{l0K} and A_s are stable. This is equivalent to when the Lyapunov equations (25), (28) and (29) have unique positive definite solutions. (When $l = 1$, the resultant closed-loop system is stable if and only if A_{sK} and A_{l0K} are stable.)

When the actuators of l subsystems outages occur, the performance index is

$$J = x_0^{\mathrm{T}} W_l x_0,$$

where W_l can be computed by solving the Lyapunov equation

$$(A + BN_l\tilde{K})^{\mathrm{T}} W_l + W_l(A + BN_l\tilde{K}) + Q + \tilde{K}^{\mathrm{T}} N_l^{\mathrm{T}} R N_l \tilde{K} = 0,$$

where

$$N_l = \mathrm{diag}\left[\overbrace{0, \ldots, 0}^{l}, \overbrace{I, \ldots, I}^{N-l} \right].$$

From lemma 4,

$$P^{-1} W_l P = \mathrm{diag}\left[W_{l0K}, \overbrace{W_s, \ldots, W_s}^{l-1}, \overbrace{W_{sK}, \ldots, W_{sK}}^{N-l-1} \right]$$

where W_{l0K}, W_s and W_{sK} are the solutions of (28), (29) and (25) respectively. Hence $J \leqslant J_0$ if and only if (27) holds. The proof is completed. □

Remark 2: Although theorem 1 gives a necessary and sufficient condition for a controller of the form (23) to be a reliable controller. There is still no systematic method to choose $\tilde{K}_1$ and $\tilde{K}_2$ such that the conditions (P1) and (P2) hold. However, we suggest using the following method to choose $\tilde{K}_1$ and $\tilde{K}_2$.

Suppose that Ω contains the actuators of the first l_0 subsystems. Use the results of Veillete (1995) to construct a reliable LQ controller which can tolerate outages of actuators in Ω. From lemma 4, the reliable LQ state-feedback gain matrix K should have the form

$$K = \begin{bmatrix} \bar{S}(K_1, K_{11}, l_0) & \tilde{S}(K_{12}, K_{12}, l_0, N - l_0) \\ \tilde{S}(K_{21}, K_{21}, N - l_0, l_0) & \bar{S}(K_2, K_{22}, N - l_0) \end{bmatrix}, \tag{30}$$

where $K_1, K_{11}, K_{12}, K_{21}, K_2, K_{22} \in \mathbb{R}^{m \times n}$. Since K_2 and K_{22} play key roles in the reliable controller (30), we suggest choosing

$$\tilde{K}_1 = K_2, \quad \tilde{K}_2 = K_{22},$$

to construct a controller of the form (23), and then using theorem 1 to verify its fault tolerance. □

Above, we have considered the special case that the fact that actuator outages occur in l subsystems means that all the actuators of l subsystems fail simultaneously. Now we consider a more general case. Suppose that $m > 1$, decompose the control inputs u_i as

$$u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} \quad (i = 1, \ldots, N),$$

where $u_{i1} \in \mathbb{R}^{m_1}$ denotes the subset of actuators in the ith subsystem which are prone to failure, and $u_{i2} \in \mathbb{R}^{m_2}$ $(m_1 + m_2 = m)$ denotes the subset of actuators in the ith subsystem which are not prone to failure. Furthermore, the fact that actuator outages occur in the ith subsystem means that the actuators in u_{i1} fail.

The matrices $B_1, \tilde{K}_1$ and $\tilde{K}_2$ are partitioned conformally with u_{i1} and u_{i2} as

$$B_1 = [B_{11}, B_{12}],$$
$$\tilde{K}_1 = \begin{bmatrix} \tilde{K}_{11} \\ \tilde{K}_{12} \end{bmatrix}, \quad \tilde{K}_2 = \begin{bmatrix} \tilde{K}_{21} \\ \tilde{K}_{22} \end{bmatrix},$$
$$R_1 = \mathrm{diag}[R_{11}, R_{12}].$$

The following theorem can be proved using a similar method to that of theorem 1.

Theorem 2: *Suppose that the performance bound is J_0 and the initial state of the system is x_0, then the controller of the form* (23) *is a reliable controller which can tolerate l_0 $(1 \leqslant l_0 \leqslant N)$ subsystems faults in u_{i1} if and only if $\tilde{K}_1$ and $\tilde{K}_2$ satisfy condition* (P1) *and the following condition.*

(PP2) *For $1 \leqslant l \leqslant l_0$,*

$$x_0^{\mathrm{T}} P \,\mathrm{diag}\Big[\tilde{W}_{l0K}, \overbrace{\tilde{W}_{sK}, \ldots, \tilde{W}_{sK}}^{l-1}, \overbrace{W_{sK}, \ldots, W_{sK}}^{N-l-1}\Big] P^{-1} x_0 \leqslant J_0,$$

where P is the orthogonal matrix in lemma 3, W_{sK} is the solution of (25) *and $\tilde{W}_{l0K}$ and $\tilde{W}_{sK}$ are the unique positive definite solutions of the Lyapunov equations*

$$\tilde{A}_{l0K}^{\mathrm{T}} \tilde{W}_{l0K} + \tilde{W}_{l0K} \tilde{A}_{l0K} + \mathrm{diag}[Q_1, Q_1] + \begin{bmatrix} \tilde{K}_{l0}^{11} & \tilde{K}_{l0}^{12} \\ (\tilde{K}_{l0}^{12})^{\mathrm{T}} & \tilde{K}_{l0}^{22} \end{bmatrix} = 0$$

and

$$\tilde{A}_{sK}^{\mathrm{T}} \tilde{W}_{sK} + \tilde{W}_{sK} \tilde{A}_{sK} + Q_1 + (\tilde{K}_{12}^{\mathrm{T}} - \tilde{K}_{22}^{\mathrm{T}}) R_1 (\tilde{K}_{12} - \tilde{K}_{22}) = 0$$

respectively, where

$$\tilde{A}_{sK} = A_s + B_{12}(\tilde{K}_{12} - \tilde{K}_{22}),$$

$$\tilde{A}_{l0K} = A_{l0} + \begin{bmatrix} B_{12}[\tilde{K}_{12} + (l-1)\tilde{K}_{22}] & [l(N-l)]^{1/2} B_{12} \tilde{K}_{22} \\ [l(N-l)]^{1/2} B_1 \tilde{K}_2 & B_1[\tilde{K}_1 + (N-l-1)\tilde{K}_2] \end{bmatrix},$$

$$\tilde{K}_{l0}^{11} = l(N-l)\tilde{K}_2^{\mathrm{T}} R_1 \tilde{K}_2 + [\tilde{K}_{12}^{\mathrm{T}} + (l-1)\tilde{K}_{22}^{\mathrm{T}}] R_1 [\tilde{K}_{12} + (l-1)\tilde{K}_{22}],$$

$$\tilde{K}_{l0}^{12} = [l(N-l)]^{1/2} \tilde{K}_2^{\mathrm{T}} R_1 [\tilde{K}_1 + (N-l-1)\tilde{K}_2] + [l(N-l)]^{1/2} [\tilde{K}_{12}^{\mathrm{T}} + (l-1)\tilde{K}_{22}^{\mathrm{T}}] R_1 \tilde{K}_{22},$$

$$\tilde{K}_{l0}^{22} = [\tilde{K}_1^{\mathrm{T}} + (N-l-1)\tilde{K}_2^{\mathrm{T}}] \times R_1 [\tilde{K}_1 + (N-l-1)\tilde{K}_2] + l(N-l)\tilde{K}_{22}^{\mathrm{T}} R_1 \tilde{K}_{22}.$$

When $l_0 = N$, we use Ω_0 to denote all the actuators in u_{i1} $(i = 1, \ldots, N)$. The designed controller should tolerate all the actuator outages within Ω_0. For this particular problem, using the special structure of system (1) and the results of Veillette (1995), we can obtain the following theorem.

Theorem 3: *Suppose that $\bar{X}_s$ and $\bar{X}_t$ are the unique positive definite solutions of algebraic Riccati equations*

$$A_s^{\mathrm{T}} \bar{X}_s + \bar{X}_s A_s - \bar{X}_s B_{12} R_{12}^{-1} B_{12}^{\mathrm{T}} \bar{X}_s + Q_1 = 0 \tag{31}$$

and

$$A_t^{\mathrm{T}} \bar{X}_t + \bar{X}_t A_t - \bar{X}_t B_{12} R_{12}^{-1} B_{12}^{\mathrm{T}} \bar{X}_t + Q_1 = 0 \qquad (32)$$

respectively, then the reliable state-feedback gain matrix $\tilde{K}$ *can be constructed by* (23) *where*

$$\tilde{K}_1 = -\frac{R_1^{-1} B_1^{\mathrm{T}} [\bar{X}_t + (N-1)\bar{X}_s]}{N},$$

$$\tilde{K}_2 = -\frac{R_1^{-1} B_1^{\mathrm{T}} (\bar{X}_t - \bar{X}_s)}{N}.$$

In this case, the state-feedback system remains stable and satisfies the performance bound $J \leqslant x_0^{\mathrm{T}} \bar{X} x_0$ *even if actuator outages occur within* Ω_0*, where* x_0 *is the initial state of the system, and*

$$\bar{X} = \bar{S}\left(\frac{\bar{X}_t + (N-1)\bar{X}_s}{N}, \frac{\bar{X}_t - \bar{X}_s}{N}, N\right). \qquad (33)$$

Proof: From the results of Veillette (1995), the reliable state-feedback gain matrix $\tilde{K}$ can be constructed from

$$\tilde{K} = -R^{-1} B^{\mathrm{T}} \bar{X},$$

where $\bar{X}$ is the solution of the algebraic Riccati equations

$$A^{\mathrm{T}} \bar{X} + \bar{X} A - \bar{X} B_{\Omega_0'} R_{\Omega_0'}^{-1} B_{\Omega_0'}^{\mathrm{T}} \bar{X} + Q = 0,$$

where Ω_0' denotes the subset of actuators in u_{i2} $(i = 1, \ldots, N)$. From theorem 1 of Sundareshan and Elbanna (1991), $\bar{X}$ has the structure (33), where $\bar{X}_s$ and $\bar{X}_t$ are the solutions of (31) and (32) respectively. The proof is completed. □

4.2. *Decentralized reliable controller design*

When a decentralized reliable controller is considered, the gain matrix of the controller should have the form

$$\hat{K} = \mathrm{diag}\,[\hat{K}_1, \ldots, \hat{K}_1] \in \mathbb{R}^{Nm \times Nn}. \qquad (34)$$

In this case, theorem 1 is simplified to the following corollary.

Corollary 1: *Suppose that the performance bound is* J_0*, the initial state of the system is* x_0*, then the controller of the form* (34) *is a reliable controller which can tolerate* l_0 *subsystems faults if and only if* $\hat{K}_1$ *satisfies the following conditions.*

(p1)

$$x_0^{\mathrm{T}} T_{Nn} \mathrm{diag}\left[\hat{W}_{tK}, \overbrace{\hat{W}_{sK}, \ldots, \hat{W}_{sK}}^{N-1}\right] T_{Nn}^{-1} x_0 \leqslant J_0,$$

where $\hat{W}_{tK}$ *and* $\hat{W}_{sK}$ *are the solutions of the Lyapunov equations*

$$(A_t + B_1 \hat{K}_1)^{\mathrm{T}} \hat{W}_{tK} + \hat{W}_{tK} (A_t + B_1 \hat{K}_1) + Q_1 + \hat{K}_1^{\mathrm{T}} R_1 \hat{K}_1 = 0$$

and

$$(A_s + B_1 \hat{K}_1)^{\mathrm{T}} \hat{W}_{sK} + \hat{W}_{sK} (A_s + B_1 \hat{K}_1) + Q_1 + \hat{K}_1^{\mathrm{T}} R_1 \hat{K}_1 = 0 \qquad (35)$$

respectively.

(p2) *For* $1 \leqslant l \leqslant l_0$*,*

$$x_0^{\mathrm{T}} P\, \mathrm{diag}\left[\hat{W}_{l0K}, \overbrace{W_s, \ldots, W_s}^{l-1}, \overbrace{\hat{W}_{sK}, \ldots, \hat{W}_{sK}}^{N-l-1}\right] P^{-1} x_0 \leqslant J_0,$$

where $\hat{W}_{sK}$ *is the solution of* (35)*,* W_s *is the solution of* (29)*, and* $\hat{W}_{l0K}$ *is the unique solution of the Lyapunov equation*

$$\hat{A}_{l0K}^{\mathrm{T}} \hat{W}_{l0K} + \hat{W}_{l0K} \hat{A}_{l0K} + \mathrm{diag}\,[Q_1, Q_1] + \mathrm{diag}\,[0, \hat{K}_1^{\mathrm{T}} R_1 \hat{K}_1] = 0$$

where

$$\hat{A}_{l0K} = A_{l0} + \mathrm{diag}\,[0, B_1 \hat{K}_1].$$

Similarly, if we decompose $\hat{K}_1$ conformally with u_i as

$$\hat{K}_1 = \begin{bmatrix} \hat{K}_{11} \\ \hat{K}_{12} \end{bmatrix},$$

then theorem 2 becomes the following corollary.

Corollary 2: *Suppose that the performance bound is* J_0 *and the initial state of the system is* x_0*; then the controller of the form* (34) *is a reliable controller which can tolerate* l_0 $(1 \leqslant l_0 \leqslant N)$ *subsystems faults in* u_{i1} *if and only if* $\hat{K}_1$ *satisfies condition* (p1) *and the following condition.*

(pp2) *For* $1 \leqslant l \leqslant l_0$*,*

$$x_0^{\mathrm{T}} P\, \mathrm{diag}\left[\check{W}_{l0K}, \overbrace{\check{W}_{sK}, \ldots, \check{W}_{sK}}^{l-1}, \overbrace{\hat{W}_{sK}, \ldots, \hat{W}_{sK}}^{N-l-1}\right] P^{-1} x_0 \leqslant J_0,$$

where P *is the orthogonal matrix in lemma 3,* $\hat{W}_{sK}$ *is the solution of* (35) *and* $\check{W}_{l0K}$ *and* $\check{W}_{sK}$ *are the unique positive definite solutions of the Lyapunov equations*

$$\check{A}_{l0K}^{\mathrm{T}} \check{W}_{l0K} + \check{W}_{l0K} \check{A}_{l0K} + \mathrm{diag}\,[Q_1, Q_1] + \mathrm{diag}\,[\hat{K}_{12}^{\mathrm{T}} R_1 \hat{K}_{12}, \hat{K}_1^{\mathrm{T}} R_1 \hat{K}_1] = 0$$

and

$$(A_s + B_{12}\check{K}_{12})^{\mathrm{T}}\check{W}_{sK} + \check{W}_{sK}(A_s + B_{12}\check{K}_{12}) + Q_1 + \hat{K}_{12}^{\mathrm{T}}R_1\hat{K}_{12} = 0$$

respectively, where

$$\check{A}_{l0K} = A_{l0} + \mathrm{diag}\,[B_{12}\hat{K}_{12}, B_1\hat{K}_1].$$

Remark 3: From the above corollaries, the decentralized reliable LQ controller design problem can be transformed to a quadratic matrix inequality (QMI) problem. For example, from corollary 1, a decentralized reliable LQ controller of the form (34) exists if there exist positive definite matrices $\hat{W}_{tK}, \hat{W}_{sK}, \hat{W}_{l0K}, W_s$ and matrix $\hat{K}_1$ satisfy the following matrix inequalities:

$$x_0^{\mathrm{T}} T_{Nn}\mathrm{diag}[\hat{W}_{tK}, \overbrace{\hat{W}_{sK}, \ldots, \hat{W}_{sK}}^{N-1}]T_{Nn}^{-1}x_0 < J_0$$

$$x_0^{\mathrm{T}} P\,\mathrm{diag}\left[\hat{W}_{l0K}, \overbrace{W_s, \ldots, W_s}^{l-1}, \overbrace{\hat{W}_{sK}, \ldots, \hat{W}_{sK}}^{N-l-1}\right]P^{-1}x_0 < J_0, \quad l = 1, \ldots, l_0,$$

$$(A_s + B_1\hat{K}_1)^{\mathrm{T}}\hat{W}_{sK} + \hat{W}_{sK}(A_s + B_1\hat{K}_1) + Q_1 + \hat{K}_1^{\mathrm{T}}R_1\hat{K}_1 < 0,$$

$$(A_t + B_1\hat{K}_1)^{\mathrm{T}}\hat{W}_{tK} + \hat{W}_{tK}(A_t + B_1\hat{K}_1) + Q_1 + \hat{K}_1^{\mathrm{T}}R_1\hat{K}_1 < 0,$$

$$\hat{A}_{l0K}^{\mathrm{T}}\hat{W}_{l0K} + \hat{W}_{l0K}\hat{A}_{l0K} + \mathrm{diag}\,[Q_1, Q_1] + \mathrm{diag}\,[0, \hat{K}_1^{\mathrm{T}}R_1\hat{K}_1] < 0, \quad l = 1, \ldots, l_0,$$

$$A_s^{\mathrm{T}}W_s + W_sA_s + Q_1 < 0.$$

This problem becomes a simultaneous LQ control problem and may be solved by an iterative linear matrix inequality (LMI) approach (Lam and Cao 1999). □

Remark 4: It should be noted that the symmetry structure of the controllers (23) and (34) stems from the symmetry structure of symmetric composite systems and are used to retain the structural properties of the systems. □

5. Example

Consider a symmetric composite system given by (1) and (2). Suppose that $N = 5$, $m = n = 2$ and

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Computing directly, we have

$$\mathrm{spec}\,(A_t) = \{2, -7\} \nsubseteq \mathbb{C}^-.$$

Hence the open-loop system is unstable.

Now we design reliable LQ controllers for the above system. We first suppose that $l_0 = 2$ and $J_0 = 200$, using the design procedure suggested in remark 2 to construct the centralized reliable controller and obtain

$$u = \bar{S}(K_2, K_{22}, 5)x, \tag{36}$$

where

$$K_2 = \begin{bmatrix} -0.6593 & -0.8322 \\ -0.8935 & -0.6593 \end{bmatrix},$$

$$K_{22} = \begin{bmatrix} -0.6892 & -0.6701 \\ -0.6443 & -0.6892 \end{bmatrix}.$$

Assume the initial state $x_0 = (1,1,1,1,1,1,1,1,1,1)^{\mathrm{T}}$, when no actuator outage occurs, the performance index is $J = 49.6379$. The performance indices of several kinds of actuator outages are given in table 1. Table 1 shows that the closed-loop system remains stable and satisfies the performance index bound J_0 in spite of the outages of actuators in *three* subsystems. The system becomes unstable when the outages of actuators in *four* subsystems occurred. Thus controller (36) is a reliable LQ controller as required.

When a decentralized controller is considered, we use the method in remark 3 for its construction. Using an iterative LMI method to solve the QMIs in remark 3, we obtain

$$\hat{K}_1 = \begin{bmatrix} -6.7185 & -20.6403 \\ -23.2924 & -6.5984 \end{bmatrix}.$$

Hence the controller

$$u = \mathrm{diag}\,[\hat{K}_1, \ldots, \hat{K}_1]x \tag{37}$$

is a decentralized reliable LQ controller as required. The performance indices of several kinds of actuator outages are given in table 2.

Now we define $u_i = [\,u_{i1}, u_{i2}]$ $(i = 1, \ldots, 5, u_{i1}, u_{i2} \in \mathbb{R})$ and suppose that the u_{i1} are prone to failure. When

Table 1. Performance indices under controller (36).

Outages of u_i	0	1	2	3	4	5
J	49.6379	56.0805	71.1771	148.3108	(Unstable)	(Unstable)

Table 2. Performance indices under controller (37).

Outages of u_i	0	1	2	3	4	5
J	154.2899	131.2265	119.9903	(Unstable)	(Unstable)	(Unstable)

Table 3. Performance indices under controller (38).

Outages of u_{i1}	0	1	2	3	4	5
J	55.3757	57.6240	60.6478	64.9344	71.4902	82.7859

$l_0 = N = 5$, theorem 3 is used to obtain the reliable controller

$$u = \bar{S}(\tilde{K}_1, \tilde{K}_2, 5)x, \tag{38}$$

where

$$\tilde{K}_1 = \begin{bmatrix} -0.7954 & -0.9731 \\ -1.0303 & -0.7954 \end{bmatrix},$$

$$\tilde{K}_2 = \begin{bmatrix} -0.8271 & -0.8066 \\ -0.7801 & -0.8271 \end{bmatrix}.$$

Assume the same initial state x_0 as before, the state-feedback system remains stable and satisfies the performance bound $J \leqslant 82.7859$ despite the outages of any u_{i1}. The performance indices of several kinds of actuator outage are given in table 3.

6. Conclusion

In this paper, we first proved that the solution of a special class of Riccati equations can be constructed by the solutions of three much-lower-order Riccati equations. As an application of this result, we gave some simple methods to design the reliable LQ state feedback regulator for symmetric composite systems. The designed regulator has the property that guaranteed the stability and the performance bound of the closed-loop system despite the fact that actuator outages occur.

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References

Bakule, L., and Rodellar, J., 1996, Decentralized control design of uncertain nominally symmetric composite systems. *Proceedings of the Institution of Electrical Engineers*, Pt D, **143,** 530–535.

Hovd, M., Braatz, R., and Skogestad, S., 1997, SVD controllers for H_2-, H_∞- and μ-optimal control. *Automatica*, **33,** 433–439.

Hovd, M., and Skogestad, S., 1994, Control of symmetrically interconnected plants. *Automatica*, **30,** 957–973.

Huang, S., Lam, J., Yang, G. H., and Zhang, S., 1999, Fault tolerant decentralized H_∞ control for symmetric composite systems. *IEEE Transactions on Automatic Control*, **44,** 2108–2114.

Lam, J., and Cao, Y. Y., 1999, Simultaneous linear-quadratic optimal control design via static output feedback. *International Journal of Robust and Nonlinear Control*, **9,** 551–558.

Lam, J., and Yang, G. H., 1996, Balanced model reduction of symmetric composite systems. *International Journal of Control*, **65,** 1031–1043.

Liu, X., 1992, Output regulation of strongly coupled symmetric composite systems. *Automatica*, **28,** 1037–1041.

Lunze, J., 1986, Dynamics of strongly coupled symmetric composite systems. *International Journal of Control*, **44,** 1617–1640.

Sundareshan, M. K., and Elbanna, R. M., 1991, Qualitative analysis and decentralized controllers synthesis for a class of large-scale systems with symmetrically interconnected subsystems. *Automatica*, **27,** 383–388.

Veillette, R. J., 1995, Reliable linear–quadratic state-feedback control. *Automatica*, **31,** 137–143.

Veillette, R. J., Medanic, J. V., and Perkins, W. R., 1992, Design of reliable control systems. *IEEE Transactions on Automatic Control*, **37,** 290–304.

Yang, G. H., Wang, J., Soh, C. B., and Lam, J., 1996, Design of reliable controllers for symmetric composite systems: primary contingency case. *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, 1996, pp. 3612–3617.

Yang, G. H., and Zhang, S. Y., 1995, Stabilizing controllers for uncertain symmetric composite systems. *Automatica*, **31,** 337–340.

Yang, G. H., Zhang, S. Y., Lam, J., and Wang, J., 1998, Reliable control using redundant controllers. *IEEE Transactions on Automatic Control*, **43,** 1588–1593.