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Saturated linear quadratic regulation of uncertain linear systems: stability region estimation and controller design

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This paper considers the problems of estimating the stability region (domain of attraction) and controller design for uncertain linear continuous-time systems with input saturation when linear quadratic (LQ) optimal controller is used. By exploiting the structure of the LQ controller and the property of saturation functions, it is established that the estimation of stability region can be obtained by solving linear matrix inequality (LMI) problems. Moreover, an iterative LMI (ILMI) algorithm is presented to design an LQ controller such that the largest estimated stability region can be obtained. Two examples are given to compare our results with existing ones.

1. Introduction

Input saturation is a common feature of control systems. The stabilization problem of linear systems with input saturation has been studied by many researchers (see, e.g. Bernstein and Michel 1995, Fossard and Normand-Cyrot 1996, Tarbouriech and Garcia 1997 and the references therein). For linear continuous-time systems with some poles in the open right-half-plane, globally asymptotically stabilization and semiglobally asymptotically stabilization using saturated feedback controllers are both impossible (Sussmann *et al.* 1994, Saberi *et al.* 1996). In other words, we can only obtain locally asymptotically stabilization properties. These bring about two important problems for open-loop unstable linear systems with input saturation. The first one is to estimate the *stability region* (*domain of attraction*) for the closed-loop system with a given saturating controller. The second one is to design a controller in order to obtain a larger *stability region*. In the last few years, different methods have been used to study these two problems, not only for systems with no uncertainties (Tyan and Bernstein 1995, Barbu *et al.* 1997, Pittet *et al.* 1997, Hindi and Boyd 1998, Gomes da Silva Jr and Tarbouriech 1999 a, Pare *et al.* 1999, Scibile and Kouvaritakis 2000), but also for uncertain systems (Kim and Bien 1994, Henrion and Tarbouriech 1999, Henrion *et al.* 1999).

Since linear quadratic (LQ) controllers play significant parts in control theory and applications, we study the above two problems for uncertain linear continuous-time systems with input saturation when LQ controllers are used. Both unstructured uncertainties and structured uncertainties are considered in this paper. By investigating the property of the saturation functions and exploiting the structure of the controllers, we give certain linear matrix inequality (LMI) methods to estimate an ellipsoidal-type stability region. For the controller design problem, we prove that an LQ controller that leads to the largest estimated stability region can be obtained by solving a matrix inequality (MI) optimization problem. Since it is a difficult task to find the true optimal value of the non-linear MI problem, an iterative LMI (ILMI) algorithm is presented to obtain an approximated solution of the MI problem. The results in this paper are compared with that of Kim and Bien (1994), Pittet *et al.* (1997), Henrion and Tarbouriech (1999), Henrion *et al.* (1999) and Gomes da Silva Jr and Tarbouriech (1999 a) using two examples.

This paper is organized as follows. Section 2 provides preliminaries and problem statements. The estimation methods of stability region are presented in § 3. In § 4, an ILMI method is given to design a controller which achieves the largest estimated stability region. The methods are illustrated in § 5 with two examples borrowed from Kim and Bien (1994) and Henrion *et al.* (1999). Section 6 concludes the paper.

2. Preliminaries and problem statements

The following notations are used in this paper. e_i denotes the ith standard basis of dimension m. $\|\cdot\|$ denotes the Euclidean norm of a vector. For a symmetric matrix M, $M > (\geq)\, 0$ means that M is positive (semi-)definite, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimal and maximal eigenvalues of M, respectively. For a matrix E with $E^{\mathrm{T}}E > 0$, $\kappa(E)$ denotes the spectral condition number of E. If not explicitly stated, I and 0 denote the identity matrix and the zero matrix of appropriate dimensions, respectively. Besides, all matrices are assumed to have compatible dimensions.

Consider the uncertain linear system $\mathcal{S}$ with input saturation

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$$\mathcal{S}: \quad \dot{x} = (A + \Delta A)x + (B + \Delta B)f(u) \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are the state vector and the input vector, respectively. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ with (A, B) is stabilizable and B has no zero column. ΔA and ΔB are uncertain matrices. $f: \mathbb{R}^m \to \mathbb{R}^m$ is an m-dimensional saturation function with $f(u) = [\sigma_1(u_1), \ldots, \sigma_m(u_m)]^{\mathrm{T}}$ and $\sigma_i: \mathbb{R} \to \mathbb{R}$ are scalar saturation functions defined by

$$\sigma_i(s) = \operatorname{sign}(s)\min\{|s|, \Delta_i\}, \qquad \forall s \in \mathbb{R}, \quad (\Delta_i > 0, i = 1, \ldots, m) \tag{2}$$

In this paper, suppose $Q \in \mathbb{R}^{n\times n}$, $Q > 0$, $R = \operatorname{diag}[r_1, \ldots, r_m] \in \mathbb{R}^{m\times m}$, $R > 0$ and $P > 0$ is the unique solution of the algebraic Riccati equation

$$A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q = 0 \tag{3}$$

For $\mathcal{S}$, we can design a linear quadratic optimal controller given by

$$u = -R^{-1}B^{\mathrm{T}}Px \tag{4}$$

and the resulting closed-loop system becomes

$$\mathcal{S}_c: \quad \dot{x} = (A + \Delta A)x + (B + \Delta B)f(-R^{-1}B^{\mathrm{T}}Px) \tag{5}$$

This paper will discuss the following two problems.

Problem ESTIMATION: For a given LQ controller u, estimate the stability region of $\mathcal{S}_c$. That is, to find a set $\Omega \subset \mathbb{R}^n$ $(0 \in \Omega)$ of initial conditions such that $\forall x_0 \in \Omega$, the resulting trajectories of $\mathcal{S}_c$ asymptotically converge to 0.

Problem SYNTHESIS: Find a controller of the form (4) (that is, find Q and R) such that the above estimated stability region of the resultant closed-loop system is the largest.

3. Estimation of stability region

This section considers Problem ESTIMATION. We first prove a lemma about saturation functions.

Lemma 1: *Suppose $f: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is the m-dimensional saturation function in $\mathcal{S}$, then for any $W = \operatorname{diag}[w_1, w_2, \ldots, w_m] \in \mathbb{R}^{m\times m}$ with $2I \geq W \geq 0, R = \operatorname{diag}[r_1, r_2, \ldots, r_m] \in \mathbb{R}^{m\times m} > 0$, and $y = [y_1, y_2, \ldots, y_m]^{\mathrm{T}} \in \mathbb{R}^m$, we have*

$$\forall i \in J_W, \quad \left|\frac{y_i}{r_i}\right| \leq \frac{2\Delta_i}{w_i} \quad \Longrightarrow \quad 2y^{\mathrm{T}}f(R^{-1}y) \geq y^{\mathrm{T}}WR^{-1}y$$

where

$$J_W = \{1 \leq i \leq m: w_i > 0\} \subset \{1, \ldots, m\} \tag{6}$$

Proof: From (2), $\forall i = 1, \ldots, m$, $\forall r_i > 0$, we have the following conclusions.

(i) If $w_i = 0$, then

$$\forall y_i \in \mathbb{R} \quad \Longrightarrow \quad 2y_i\sigma_i\left(\frac{y_i}{r_i}\right) \geq \frac{w_i}{r_i}y_i^2$$

(iia) If $2 \geq w_i > 0$, then

$$\left|\frac{y_i}{r_i}\right| \leq \Delta_i \quad \Longrightarrow \quad 2y_i\sigma_i\left(\frac{y_i}{r_i}\right) = 2y_i\frac{y_i}{r_i} \geq \frac{w_i}{r_i}y_i^2$$

(iib) If $2 \geq w_i > 0$, then

$$\Delta_i \leq \left|\frac{y_i}{r_i}\right| \leq \frac{2\Delta_i}{w_i} \quad \Longrightarrow \quad 2y_i\sigma_i\left(\frac{y_i}{r_i}\right) = 2|y_i|\Delta_i \geq \frac{w_i}{r_i}y_i^2$$

From (iia) and (iib), we have

(ii) If $2 \geq w_i > 0$, then

$$\left|\frac{y_i}{r_i}\right| \leq \frac{2\Delta_i}{w_i} \quad \Longrightarrow \quad 2y_i\sigma_i\left(\frac{y_i}{r_i}\right) \geq \frac{w_i}{r_i}y_i^2$$

When

$$\frac{y_i}{r_i} \geq 0, \qquad 2y_i\sigma_i\left(\frac{y_i}{r_i}\right) \geq \frac{w_i}{r_i}y_i^2 \quad \text{becomes} \quad 2\sigma_i\left(\frac{y_i}{r_i}\right) \geq w_i\frac{y_i}{r_i}$$

and conclusion (ii) can be depicted by figure 1.

From (i) and (ii), if

$$\forall i \in J_W, \quad \left|\frac{y_i}{r_i}\right| \leq \frac{2\Delta_i}{w_i}$$

then

$$\begin{aligned} 2y^{\mathrm{T}}f(R^{-1}y) &= 2[y_1, y_2, \ldots, y_m] \\ &\quad \times \left[\sigma_1\left(\frac{y_1}{r_1}\right), \sigma_2\left(\frac{y_2}{r_2}\right), \ldots, \sigma_m\left(\frac{y_m}{r_m}\right)\right]^{\mathrm{T}} \\ &\geq \frac{w_1}{r_1}y_1^2 + \frac{w_2}{r_2}y_2^2 + \cdots + \frac{w_m}{r_m}y_m^2 \\ &= y^{\mathrm{T}}WR^{-1}y \end{aligned}$$

The proof is completed. □

The following lemma is also needed in this paper.

Lemma 2 (Franklin 1969)**:** *For any matrices Z_1 and Z_2 with appropriate dimensions, we have*

$$Z_1^{\mathrm{T}}Z_2 + Z_2^{\mathrm{T}}Z_1 \leq \gamma Z_1^{\mathrm{T}}Z_1 + \frac{1}{\gamma}Z_2^{\mathrm{T}}Z_2, \forall \gamma > 0$$

3.1. *Systems with unstructured (norm-bounded) uncertainties*

In this section, we assume the norm bound of the uncertainties in $\mathcal{S}$ are known and

$$\|\Delta A\| \leq \alpha, \qquad \|\Delta B\| \leq \beta \tag{7}$$

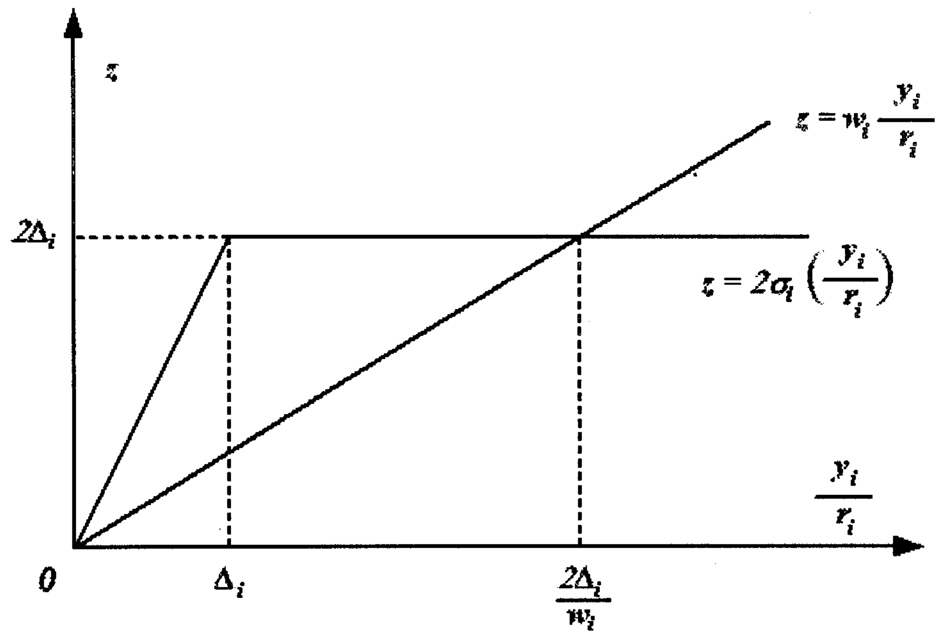

Figure 1. Idea in proof of Lemma 1.

From Lemmas 1 and 2 and the structure of the LQ controller (4), we obtain an estimation of the stability region of $\mathcal{S}_c$, as given by the following lemma.

Lemma 3: *Suppose* $\gamma_1, \gamma_2 \in \mathbb{R}$ *and* $W = \mathrm{diag}\,[w_1, w_2, \ldots, w_m] \in \mathbb{R}^{m\times m}$ *are feasible solutions of the following linear matrix inequality (LMI) problem*

$$\begin{bmatrix} -A^{\mathrm T}P - PA - (\gamma_1+\gamma_2)P^2 + PBWR^{-1}B^{\mathrm T}P & \alpha I & \beta PBR^{-1} \\ \alpha I & \gamma_1 I & 0 \\ \beta R^{-1}B^{\mathrm T}P & 0 & \gamma_2 I \end{bmatrix} > 0 \qquad (8)$$

$$W \le 2I \qquad (9)$$

$$W \ge 0 \qquad (10)$$

then

$$\Omega_W = \{x \in \mathbb{R}^n : x^{\mathrm T}Px \le V_W\} \qquad (11)$$

is contained in the stability region of $\mathcal{S}_c$, *where*

$$V_W = \min_{i\in J_W} \frac{4r_i^2\Delta_i^2}{w_i^2 e_i^{\mathrm T}B^{\mathrm T}TBe_i} \qquad (12)$$

and J_W *is defined in* (6).

Proof: Consider the derivative of the Lyapunov function $V(x) = x^{\mathrm T}Px$ along the trajectories of $\mathcal{S}_c$, from Lemma 2 and (7), for any $\gamma_1, \gamma_2 > 0$

$$\begin{aligned} \dot V(x) &= \dot x^{\mathrm T}Px + x^{\mathrm T}P\dot x \\ &= x^{\mathrm T}(PA + A^{\mathrm T}P)x - 2x^{\mathrm T}PBf(R^{-1}B^{\mathrm T}Px) \\ &\quad + 2x^{\mathrm T}P\Delta Ax - 2x^{\mathrm T}P\Delta Bf(R^{-1}B^{\mathrm T}Px) \\ &\le x^{\mathrm T}(PA + A^{\mathrm T}P)x - 2x^{\mathrm T}PBf(R^{-1}B^{\mathrm T}Px) \qquad (13) \\ &\quad + x^{\mathrm T}\left[(\gamma_1+\gamma_2)P^2 + \frac{1}{\gamma_1}\alpha^2 I\right]x \\ &\quad + \frac{1}{\gamma_2}\beta^2 f^{\mathrm T}(R^{-1}B^{\mathrm T}Px)f(R^{-1}B^{\mathrm T}Px) \end{aligned}$$

Because

$$f^{\mathrm T}(R^{-1}B^{\mathrm T}Px)f(R^{-1}B^{\mathrm T}Px) \le (x^{\mathrm T}PBR^{-1})(R^{-1}B^{\mathrm T}Px) \qquad (14)$$

we have

$$\begin{aligned} \dot V(x) \le x^{\mathrm T}\Big[&PA + A^{\mathrm T}P + (\gamma_1+\gamma_2)P^2 + \frac{1}{\gamma_1}\alpha^2 I \\ &+ \frac{1}{\gamma_2}\beta^2 PBR^{-2}B^{\mathrm T}P\Big]x - 2x^{\mathrm T}PBf(R^{-1}B^{\mathrm T}Px) \end{aligned} \qquad (15)$$

For any $W = \mathrm{diag}[w_1, w_2, \ldots, w_m]$ with $2I \ge W \ge 0$, denote

$$\mathcal{D}_W = \left\{x \in \mathbb{R}^n : \left|e_i^{\mathrm T}R^{-1}B^{\mathrm T}Px\right| \le \frac{2\Delta_i}{w_i},\ \forall i \in J_W\right\} \qquad (16)$$

from Lemma 1, if $x \in \mathcal{D}_W$, then

$$2x^{\mathrm{T}}PBf(R^{-1}B^{\mathrm{T}}Px) \geq x^{\mathrm{T}}PBWR^{-1}B^{\mathrm{T}}Px$$

and

$$\dot{V}(x) \leq x^{\mathrm{T}}\left[PA + A^{\mathrm{T}}P - PBWR^{-1}B^{\mathrm{T}}P + (\gamma_1+\gamma_2)P^2 + \frac{1}{\gamma_1}\alpha^2 I + \frac{1}{\gamma_2}\beta^2 PBR^{-2}B^{\mathrm{T}}P\right]x$$

Since γ_1, γ_2 and W satisfying LMIs (8)–(10), we have $\gamma_1, \gamma_2 > 0$, $2I \geq W \geq 0$ and

$$PA + A^{\mathrm{T}}P - PBWR^{-1}B^{\mathrm{T}}P + (\gamma_1+\gamma_2)P^2 + \frac{1}{\gamma_1}\alpha^2 I + \frac{1}{\gamma_2}\beta^2 PBR^{-2}B^{\mathrm{T}}P < 0 \tag{17}$$

Hence

$$x \in \mathcal{D}_W, \qquad x \neq 0 \qquad \Longrightarrow \qquad \dot{V}(x) < 0 \tag{18}$$

However, $\mathcal{D}_W$ may not be contained in the stability region of $\mathcal{S}_c$ because the trajectories starting from $\mathcal{D}_W$ may leave $\mathcal{D}_W$ and $\dot{V}(x) < 0$ is not guaranteed. But if we denote

$$V_i = \min\left\{x^{\mathrm{T}}Px : x \in \mathbb{R}^n,\ \left|e_i^{\mathrm{T}}R^{-1}B^{\mathrm{T}}Px\right| = \frac{2\Delta_i}{w_i}\right\}, \quad \forall i \in J_W$$

and

$$V_W = \min_{i \in J_W} V_i$$

then the set Ω_W defined by (11) is the largest ellipsoid contained in $\mathcal{D}_W$ and is contained in the stability region of $\mathcal{S}_c$.

From the fact about quadratic functions (see, e.g. Lemma 1 in Hindi and Boyd 1998), V_i can be calculated by

$$V_i = \min\left\{x^{\mathrm{T}}Px : x \in \mathbb{R}^n,\ e_i^{\mathrm{T}}B^{\mathrm{T}}Px = \frac{2r_i\Delta_i}{w_i}\right\} = \frac{(2r_i\Delta_i/w_i)^2}{(e_i^{\mathrm{T}}B^{\mathrm{T}}P)P^{-1}(PBe_i)} = \frac{4r_i^2\Delta_i^2}{w_i^2 e_i^{\mathrm{T}}B^{\mathrm{T}}PBe_i}$$

Hence V_W can be obtained by (12) and the proof is completed. □

Remark 1: In Saberi *et al*. (1996), the sufficient condition for $\dot{V}(x) < 0$ used is $\|R^{-1}B^{\mathrm{T}}Px\| \leq \Delta$ with $R = I$ and $\Delta = \Delta_1 = \cdots = \Delta_m$. In Lemma 3, by exploiting the structure of the LQ controller and the degrees of freedom in the diagonal matrix W, a less conservative sufficient condition (18) for $\dot{V}(x) < 0$ is obtained. However, the problem considered in Saberi *et al*. (1996) is a semi-global stabilization problem and hence the conservativeness of the condition is not the main issue.

Remark 2: The condition for Lemma 3 to be applicable is that LMI (8) is feasible for $W = 2I$. This condition is not strong since (3) implies the feasibility of LMI (8) for $W \geq I$ and $\alpha = \beta = 0$.

Remark 3: If LMI (8) is feasible for $W = 0$, then J_W is empty and $\Omega_W = \mathcal{D}_W = \mathbb{R}^n$. That means $\mathcal{S}_c$ is globally asymptotically stable. In this case, A must be stable because we have $A^{\mathrm{T}}P + PA < 0$ from LMI (8) with $W = 0$. For ANCBI (Asymptotically Null-Controllable with Bounded Input) systems, LMI (8) may not be feasible for $W = 0$ because of the following two reasons: (1) There are uncertainties in $\mathcal{S}$; (2) ANCBI systems can be globally stabilized by non-linear controllers (Sussmann *et al*. 1994) but may not be globally stabilized by linear controllers (Sussmann and Yang 1991).

Remark 4: For ANCBI systems with $\Delta A = \Delta B = 0$, a semi-global stabilization result can be obtained from Lemma 3. In fact, we can choose $R = I$, $Q = \varepsilon I$ and $W = I$. From Lemma 1 in Saberi *et al*. (1996), the solution P_ε of the Riccati equation

$$A^{\mathrm{T}}P_\varepsilon + P_\varepsilon A - P_\varepsilon BB^{\mathrm{T}}P_\varepsilon + \varepsilon I = 0$$

satisfies $\lim_{\varepsilon \to 0} P_\varepsilon = 0$. Since LMI (8) becomes

$$A^{\mathrm{T}}P_\varepsilon + P_\varepsilon A - P_\varepsilon BB^{\mathrm{T}}P_\varepsilon < 0$$

Lemma 3 can be used to obtain the estimated stability region

$$\Omega_{\varepsilon W} = \left\{x \in \mathbb{R}^n : x^{\mathrm{T}}P_\varepsilon x \leq V_{\varepsilon W}\right\}$$

where

$$V_{\varepsilon W} = \min_{1 \leq i \leq m} \frac{4\Delta_i^2}{e_i^{\mathrm{T}}B^{\mathrm{T}}P_\varepsilon Be_i}$$

It is easy to see that $\Omega_{\varepsilon W}$ can be made as large as one wants because $\lim_{\varepsilon \to 0} P_\varepsilon = 0$ and $\lim_{\varepsilon \to 0} V_{\varepsilon W} = +\infty$.

In Lemma 3, the estimated stability region Ω_W is dependent on W that satisfies LMIs (8)–(10). Now we consider how to choose W such that Ω_W is the largest. From Remark 3, we only need to consider the case when LMI (8) does not hold for $W = 0$, that is, J_W is nonempty.

Firstly, it should be noted that one should choose w_is as small as possible while satisfying LMIs (8)–(10).

However, it is the V_W given by (12) that determines the size of Ω_W. If $J_W \neq \{1, \ldots, m\}$, we can obtain a larger (at least not smaller) Ω_W by increasing the w_is for $i \in \{1, \ldots, m\} \backslash J_W$ and allow the decreasing (at least not increasing) of w_is for $i \in J_W$ (thereby increasing V_W). So, we can suppose

$$J_W = \{1, \ldots, m\} \tag{19}$$

Remark 5: When $J_W \neq \{1, \ldots, m\}$, the set $\mathcal{D}_W$ defined in (16) is unbounded along certain directions (there is no constraint on $\left|e_i^{\mathrm{T}} R^{-1} B^{\mathrm{T}} P x\right|$ for $i \in \{1, \ldots, m\} \backslash J_W$). However, though $\dot{V}(x) < 0$ in $\mathcal{D}_W \backslash \{0\}$, $\mathcal{D}_W$ may not be contained in the stability region of $\mathcal{S}_c$ because the trajectories starting from $\mathcal{D}_W$ may leave $\mathcal{D}_W$ subsequently. The reason why we suppose $J_W = \{1, \ldots, m\}$ is to impose some constraints on the unbounded directions of $\mathcal{D}_W$ so as to allow larger bounds on the bounded directions of $\mathcal{D}_W$ and thus obtain a larger $\Omega_W \subset \mathcal{D}_W$.

Denote

$$\Gamma_P = \operatorname{diag}\left[\frac{2\Delta_1}{\sqrt{e_1^{\mathrm{T}} B^{\mathrm{T}} P B e_1}}, \cdots, \frac{2\Delta_m}{\sqrt{e_m^{\mathrm{T}} B^{\mathrm{T}} P B e_m}}\right] \tag{20}$$

Then the following theorem holds.

Theorem 1: *Suppose* $W = \operatorname{diag}[w_1, \ldots, w_m] \in \mathbb{R}^m, \gamma_1, \gamma_2, \lambda \in \mathbb{R}$ *are the solutions of LMI problem*

$$\min \lambda \text{ subject to } (8), (9) \text{ and}$$

$$\lambda I - W R^{-1} \Gamma_P^{-1} \geq 0 \tag{21}$$

$$W > 0 \tag{22}$$

then an estimation of the stability region of $\mathcal{S}_c$ *is*

$$\Omega = \left\{x \in \mathbb{R}^n : x^{\mathrm{T}} P x \leq \frac{1}{\lambda^2}\right\} \tag{23}$$

Proof: From (19), we have (22) instead of (10). Consider the problem to maximize $V_W = \min_{1 \leq i \leq m} V_i$. That is to minimize

$$\frac{1}{\sqrt{V_W}} = \max_{1 \leq i \leq m} \frac{1}{\sqrt{V_i}}$$

Since

$$W R^{-1} \Gamma_P^{-1} = \operatorname{diag}\left[\frac{w_1 \sqrt{e_1^{\mathrm{T}} B^{\mathrm{T}} P B e_1}}{2 r_1 \Delta_1}, \cdots, \frac{w_m \sqrt{e_m^{\mathrm{T}} B^{\mathrm{T}} P B e_m}}{2 r_m \Delta_m}\right]$$
$$= \operatorname{diag}\left[\frac{1}{\sqrt{V_1}}, \ldots, \frac{1}{\sqrt{V_m}}\right]$$

the minimization of $1/\sqrt{V_W}$ is equivalent to minimizing $\lambda_{\max}(W R^{-1} \Gamma_P^{-1})$. This is further equivalent to minimizing λ that satisfies (21). Since the solution of the LMI problem is $\lambda = 1/\sqrt{V_W}$, from (11), the estimated stability region is (23). The proof is completed. □

Theorem 1 gives a method to compute the largest Ω_W for all the W satisfying the LMIs in Lemma 3. However, in the proof of Lemma 3, we have used $(x^{\mathrm{T}} P B R^{-1})(R^{-1} B^{\mathrm{T}} P x)$ as an upper bound of $f^{\mathrm{T}}(R^{-1} B^{\mathrm{T}} P x) f(R^{-1} B^{\mathrm{T}} P x)$ in (14), this may be conservative when $\left\|R^{-1} B^{\mathrm{T}} P x\right\|$ is large. Now we consider a method to obtain a possibly larger estimated stability region than Theorem 1. We first prove the following theorem.

Theorem 2: *For* $\mu > 0$, *denote*

$$G_\mu = \left\{x \in \mathbb{R}^n : \left|e_i^{\mathrm{T}} R^{-1} B^{\mathrm{T}} P x\right| < \mu \|\delta\|, \quad \forall i \in \{1, \ldots, m\}\right\}$$

where $\delta = [\Delta_1, \ldots, \Delta_m]^{\mathrm{T}}$. *If there exists* $\mu > 1$ *such that*

$$x \in G_\mu, \qquad x \neq 0 \implies \dot{V}(x) < 0 \tag{24}$$

then an estimation of the stability region of $\mathcal{S}_c$ *is given by* (23) *where* λ *is the solution of LMI problem*

$$\min \lambda \text{ subject to } (9), (21), (22) \text{ and}$$

$$\begin{bmatrix} -A^{\mathrm{T}} P - P A - (\gamma_1 + \gamma_2) P^2 + P B W R^{-1} B^{\mathrm{T}} P & \alpha I & \beta P B R^{-1} \\ \alpha I & \gamma_1 I & 0 \\ \beta R^{-1} B^{\mathrm{T}} P & 0 & \gamma_2 \mu^2 I \end{bmatrix} > 0 \tag{25}$$

Proof: Since (24) holds, we only need to consider the condition for $\dot{V}(x) < 0$ in the region

$$\mathbb{R}^n \setminus G_\mu = \left\{x \in \mathbb{R}^n : \exists i \in \{1, \ldots, m\}, \left|e_i^{\mathrm{T}} R^{-1} B^{\mathrm{T}} P x\right| \geq \mu \|\delta\|\right\}$$

In this region

$$(x^{\mathrm{T}} P B R^{-1})(R^{-1} B^{\mathrm{T}} P x) \geq \mu^2 \|\delta\|^2$$

and hence

$$f^{\mathrm{T}}(R^{-1} B^{\mathrm{T}} P x) f(R^{-1} B^{\mathrm{T}} P x) \leq \|\delta\|^2 \leq \frac{1}{\mu^2}(x^{\mathrm{T}} P B R^{-1})(R^{-1} B^{\mathrm{T}} P x)$$

From (13)

$$\dot{V}(x) \leq x^{\mathrm{T}}\left[P A + A^{\mathrm{T}} P + (\gamma_1 + \gamma_2) P^2 + \frac{1}{\gamma_1} \alpha^2 I + \frac{1}{\gamma_2} \frac{\beta^2}{\mu^2} P B R^{-2} B^{\mathrm{T}} P\right] x - 2 x^{\mathrm{T}} P B f(R^{-1} B^{\mathrm{T}} P x) \tag{26}$$

The only difference between (26) and (15) is the fifth item in the right sides of the inequalities. Hence the theorem can be proved similarly as Lemma 3 and Theorem 1. □

Remark 6: Since LMI (25) is weaker than LMI (8) only when $\mu > 1$, we have supposed $\mu > 1$ in Theorem 2. The main problem in the application of Theorem 2 is how to choose $\mu > 1$ such that (24) holds. In fact, after W is obtained by Theorem 1, we can choose

$$\mu = \min_{1\le i\le m} \frac{2\Delta_i}{w_i\|\delta\|} \tag{27}$$

then

$$G_\mu = \left\{x \in \mathbb{R}^n : \left|e_i^{\mathrm{T}} R^{-1} B^{\mathrm{T}} P x\right| < \min_{1\le i\le m} \frac{2\Delta_i}{w_i},\right.$$
$$\left.\forall i \in \{1,\dots,m\}\right\} \subset \mathcal{D}_W$$

Hence (24) holds. If $\mu > 1$, then we can apply Theorem 2 to obtain a larger estimation of the stability region. If $\mu \le 1$, then Theorem 2 is not applicable.

From Theorems 1 and 2 and Remark 6, the following procedure can be used to estimate the stability region of $\mathcal{S}_c$.

Procedure ESTIMATION:

Step 1. Set $\mu_0 = 1$, and a small tolerance scalar $\varepsilon > 0$.

Step 2. Solve the LMI problem in Theorem 1 to obtain W and an estimated stability region Ω.

Step 3. Compute μ using (27).

Step 4. If $\mu \le \mu_0 + \varepsilon$ then stop. If $\mu > \mu_0 + \varepsilon$, then solve the LMI problem in Theorem 2 to obtain a new estimated stability region Ω together with a new W.

Step 5. Let $\mu_0 = \mu$. Go to Step 3.

Remark 7: When $\Delta A = \Delta B = 0$ in $\mathcal{S}$. The estimation of the stability region of $\mathcal{S}_c$ is given by (23) where λ is the solution of LMI problem

$$\min \lambda \text{ subject to } \quad (9), (21), (22) \text{ and}$$
$$-A^{\mathrm{T}}P - PA + PBWR^{-1}B^{\mathrm{T}}P > 0 \tag{28}$$

The above LMI problem is always feasible because $W = I$ satisfies (9), (22) and (28).

3.2. *Systems with structured uncertainties*

In this section, we consider that the uncertainties in $\mathcal{S}$ admit the following structures

$$\Delta A = D_1F_1(t)E_1, \qquad \Delta B = D_2F_2(t)E_2 \tag{29}$$

where D_1, E_1, D_2, E_2 are known matrices with $E_2^{\mathrm{T}}E_2$ diagonal and

$$F_1^{\mathrm{T}}(t)F_1(t) \le I, \qquad F_2^{\mathrm{T}}(t)F_2(t) \le I \tag{30}$$

Using a similar method as in the case of unstructured uncertainties, the following corollary can be obtained.

Corollary 1: *Suppose* $W = \mathrm{diag}[w_1,\dots,w_m] \in \mathbb{R}^m, \gamma_1, \gamma_2, \lambda \in \mathbb{R}$ *is the solution of LMI problem*

$$\min \lambda \textit{ subject to } (9), (21), (22) \textit{ and}$$

$$\begin{bmatrix} -A^{\mathrm{T}}P - PA - P(\gamma_1 D_1D_1^{\mathrm{T}} + \gamma_2 D_2D_2^{\mathrm{T}})P + PBWR^{-1}B^{\mathrm{T}}P & E_1^{\mathrm{T}} & PBR^{-1}E_2^{\mathrm{T}} \\ E_1 & \gamma_1 I & 0 \\ E_2R^{-1}B^{\mathrm{T}}P & 0 & \gamma_2 I \end{bmatrix} > 0$$

then an estimation of the stability region of $\mathcal{S}_c$ *is given by* (23).

Proof: From Lemma 2 and (30), for any $\gamma_1, \gamma_2 > 0$

$$\begin{aligned}\dot V(x) &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x - 2x^{\mathrm{T}}PBf(R^{-1}B^{\mathrm{T}}Px)\\ &\quad + 2x^{\mathrm{T}}PD_1F_1(t)E_1x - 2x^{\mathrm{T}}PD_2F_2(t)E_2f(R^{-1}B^{\mathrm{T}}Px)\\ &\le x^{\mathrm{T}}\big(PA + A^{\mathrm{T}}P\big)x - 2x^{\mathrm{T}}PBf(R^{-1}B^{\mathrm{T}}Px)\\ &\quad + x^{\mathrm{T}}\left(\gamma_1 PD_1D_1^{\mathrm{T}}P + \frac{1}{\gamma_1}E_1^{\mathrm{T}}E_1\right)x\\ &\quad + \gamma_2 x^{\mathrm{T}}PD_2D_2^{\mathrm{T}}Px\\ &\quad + \frac{1}{\gamma_2}f^{\mathrm{T}}(R^{-1}B^{\mathrm{T}}Px)E_2^{\mathrm{T}}E_2f(R^{-1}B^{\mathrm{T}}Px)\end{aligned}$$

Because $E_2^{\mathrm{T}}E_2$ is diagonal and positive semi-definite, we have

$$\begin{aligned}&f^{\mathrm{T}}(R^{-1}B^{\mathrm{T}}Px)E_2^{\mathrm{T}}E_2f(R^{-1}B^{\mathrm{T}}Px)\\ &\qquad \le (x^{\mathrm{T}}PBR^{-1})E_2^{\mathrm{T}}E_2(R^{-1}B^{\mathrm{T}}Px)\end{aligned} \tag{31}$$

hence

$$\begin{aligned}\dot V(x) &\le x^{\mathrm{T}}\bigg(PA + A^{\mathrm{T}}P + \gamma_1 PD_1D_1^{\mathrm{T}}P + \frac{1}{\gamma_1}E_1^{\mathrm{T}}E_1\\ &\qquad + \gamma_2 PD_2D_2^{\mathrm{T}}P + \frac{1}{\gamma_2}PBR^{-1}E_2^{\mathrm{T}}E_2R^{-1}B^{\mathrm{T}}P\bigg)x\\ &\qquad - 2x^{\mathrm{T}}PBf(R^{-1}B^{\mathrm{T}}Px)\end{aligned}$$

The rest of the proof is similar as that of Lemma 3 and Theorem 1 and is omitted here. □

Remark 8: Very often E_2 is used to represent a scaling on each input and, in this case, E_2 itself is diagonal. The condition $E_2^{\mathrm{T}}E_2$ is diagonal is already more general in this respect by also allowing some orthogonal combination of inputs in this part of the uncertainty structure. Hence, whether restrictive or not depends upon specific application. Of course, $E_2^{\mathrm{T}}E_2$ is diagonal leads to (31) which in turn gives a better (that is, less conservative) estimation of the term $f^{\mathrm{T}}(R^{-1}B^{\mathrm{T}}Px)E_2^{\mathrm{T}}E_2f(R^{-1}B^{\mathrm{T}}Px)$. When it is not diagonal, a more conservative estimation can be used and a more conservative result than Corollary 1 can be obtained.

When $E_2^{\mathrm{T}}E_2$ is not only diagonal but also positive definite, a similar result as Theorem 2 can be obtained, as given in the following corollary.

Corollary 2: *For* $\mu > 0$, *denote*

$$\tilde{G}_\mu = \left\{x \in \mathbb{R}^n : \left|e_i^{\mathrm{T}} R^{-1} B^{\mathrm{T}} P x\right| < \mu\kappa(E_2)\|\delta\|, \quad \forall i \in \{1, \ldots, m\}\right\}$$

If there exists $\mu > 1$ *such that*

$$x \in \tilde{G}_\mu, \qquad x \neq 0 \quad \Longrightarrow \quad \dot{V}(x) < 0$$

then an estimation of the stability region of $\mathcal{S}_c$ *is given by* (23) *where* λ *is the solution of LMI problem*

$$\min \lambda \ \textit{subject to}\ (21), (9), (22)\ \textit{and}$$

$$\begin{bmatrix} -A^{\mathrm{T}}P - PA - P(\gamma_1 D_1 D_1^{\mathrm{T}} + \gamma_2 D_2 D_2^{\mathrm{T}})P + PBWR^{-1}B^{\mathrm{T}}P & E_1^{\mathrm{T}} & PBR^{-1}E_2^{\mathrm{T}} \\ E_1 & \gamma_1 I & 0 \\ E_2 R^{-1} B^{\mathrm{T}} P & 0 & \gamma_2 \mu^2 I \end{bmatrix} > 0$$

Remark 9: When $E_2^{\mathrm{T}}E_2$ is diagonal and positive definite, Procedure ESTIMATION can be used by substituting Theorem 1 with Corollary 1, substituting Theorem 2 with Corollary 2, and substituting (27) with

$$\mu = \min_{1 \le i \le m} \frac{2\Delta_i}{w_i \kappa(E_2)\|\delta\|}$$

Remark 10: When the uncertainties in $\mathcal{S}$ have the structure $[\Delta A \quad \Delta B] = DF(t)[E_1 \quad E_2]$, less conservative results may not be possible using the present approach with the conditions $D_1 = D_2$ and $F_1(t) = F_2(t)$.

Remark 11: Though we have introduced some optimization in the estimation methods in this section, our estimation may still be conservative in some cases because the shape of the region is determined by the fixed matrix P.

4. Controller design

This section considers Problem SYNTHESIS. As the controller is not known *a priori*, Theorem 2 and Corollary 2 cannot be used. So we consider the problem to find Q and R such that the estimated stability region Ω in Theorem 1 (Corollary 1) is the largest. Notice that Q does not appear in the LMIs of Theorem 1 (Corollary 1), the problem is to find P and R satisfying

$$-A^{\mathrm{T}}P - PA + PBR^{-1}B^{\mathrm{T}}P > 0 \tag{32}$$

such that the estimated stability region is the largest.

4.1. *System with unstructured uncertainties*

We first consider the case when ΔA and ΔB satisfy (7). When we consider the problem to obtain the largest *volume* of the estimated stability region Ω in Theorem 1, the following theorem is obtained.

Theorem 3: *Suppose* $X \in \mathbb{R}^{n\times n} > 0$, $\tilde{W} = \mathrm{diag}[\tilde{w}_1, \ldots, \tilde{w}_m] \in \mathbb{R}^{m\times m} > 0$, $\tilde{R} = \mathrm{diag}[\tilde{r}_1, \ldots, \tilde{r}_m] \in \mathbb{R}^{m\times m} > 0$, $\gamma_{1\lambda}, \gamma_{2\lambda} \in \mathbb{R}$ *are the solutions of the matrix inequality (MI) problem*

$$\max \det(X) \ \textit{subject to} \tag{33}$$

$$\begin{bmatrix} -XA^{\mathrm{T}} - AX - (\gamma_{1\lambda} + \gamma_{2\lambda})I + B\tilde{W}B^{\mathrm{T}} & \alpha X & \beta B\tilde{R} \\ \alpha X & \gamma_{1\lambda} I & 0 \\ \beta \tilde{R} B^{\mathrm{T}} & 0 & \gamma_{2\lambda} I \end{bmatrix} > 0 \tag{34}$$

$$-XA^{\mathrm{T}} - AX + B\tilde{R}B^{\mathrm{T}} > 0 \tag{35}$$

$$\Gamma_{X^{-1}} - \tilde{W} \geq 0 \tag{36}$$

$$2\tilde{R} - \tilde{W} \geq 0 \tag{37}$$

where

$$\Gamma_{X^{-1}} = \mathrm{diag}\left[\frac{2\Delta_1}{\sqrt{e_1^{\mathrm{T}} B^{\mathrm{T}} X^{-1} B e_1}}, \ldots, \frac{2\Delta_m}{\sqrt{e_m^{\mathrm{T}} B^{\mathrm{T}} X^{-1} B e_m}}\right]$$

then the largest estimated stability region (in volume*) is obtained as*

$$\Omega_{\max} = \left\{x \in \mathbb{R}^n : x^{\mathrm{T}} X^{-1} x \le 1\right\} \tag{38}$$

and the corresponding LQ controller is

$$u_{\max} = -\tilde{R}B^{\mathrm{T}}X^{-1}x \tag{39}$$

Proof: Since the estimated stability region in Theorem 1 is

$$\Omega = \left\{x \in \mathbb{R}^n : x^{\mathrm{T}}(\lambda^2 P)x \le 1\right\}$$

the problem is to find $P > 0, R > 0, 2I \geq W > 0$ satisfying (8), (21) and (32) such that $\det(\lambda^2 P)$ is the smallest.

Denote

$$X = \frac{P^{-1}}{\lambda^2}, \quad \tilde{W} = \frac{WR^{-1}}{\lambda^2}, \quad \tilde{R} = \frac{R^{-1}}{\lambda^2}, \quad \gamma_{1\lambda} = \frac{\gamma_1}{\lambda^2}, \quad \gamma_{2\lambda} = \frac{\gamma_2}{\lambda^2} \tag{40}$$

then (8), (21) and (32) become

$$-XA^{\mathrm{T}} - AX - (\gamma_{1\lambda} + \gamma_{2\lambda})I - \frac{1}{\gamma_{1\lambda}}\alpha^2 X^2 - \frac{1}{\gamma_{2\lambda}}\beta^2 B\tilde{R}^2 B^{\mathrm{T}} + B\tilde{W}B^{\mathrm{T}} > 0$$
$$\tilde{W}\Gamma_{X^{-1}}^{-1} \le I$$
$$-XA^{\mathrm{T}} - AX + B\tilde{R}B^{\mathrm{T}} > 0 \qquad (41)$$

and $0 < W \le 2I$ is changed into $0 < \tilde{W} \le 2\tilde{R}$.

Since inequality (41) holds if and only if (34) holds, and minimizing

$$\det(\lambda^2 P) = \det(X^{-1})$$

is equivalent to maximizing $\det(X)$, the largest estimated stability region can be obtained by (38).

The corresponding LQ controller is

$$u_{\max} = -R^{-1}B^{\mathrm{T}}Px = -\tilde{R}B^{\mathrm{T}}X^{-1}x$$

The proof is completed. □

Remark 12: Depending on applications, other objectives could be used instead of (33). For example, we can use $\max(\lambda_{\min}(X))$ to maximize the minor axis of $\Omega_{\max}$ (Gomes da Silva Jr and Tarbouriech 1999 b) or use $\max(\mathrm{trace}(X))$ to maximize the sum of the squares of the major axes of $\Omega_{\max}$ (Hindi and Boyd 1998).

The MI problem in Theorem 3 is not an LMI problem because $\Gamma_{X^{-1}}$ in (36) is non-linear in X. However, (36) becomes an LMI when $\Gamma_{X^{-1}}$ is fixed. So we suggest using the following iterative LMI (ILMI) algorithm to design the controller.

Algorithm SYNTHESIS:

Step 1. Choose $X_0 \in \mathbb{R}^{n\times n} > 0$ and a small tolerance scalar $\eta > 0$.

Step 2. Compute

$$\Gamma_{X_0^{-1}} = \mathrm{diag}\left[\frac{2\Delta_1}{\sqrt{e_1^{\mathrm{T}}B^{\mathrm{T}}X_0^{-1}Be_1}}, \ldots, \frac{2\Delta_m}{\sqrt{e_m^{\mathrm{T}}B^{\mathrm{T}}X_0^{-1}Be_m}}\right]$$

Step 3. Solve LMI problem

$$\max\ \det(X) \quad \text{subject to (34), (35), (37) and}$$

$$\Gamma_{X_0^{-1}} - \tilde{W} \ge 0 \qquad (42)$$

$$X \ge X_0 \qquad (43)$$

to obtain X.

Step 4. If $\frac{|\det(X)-\det(X_0)|}{\det(X_0)} < \eta$, go to Step 6, else go to Step 5.

Step 5. Let $X_0 = X$, go to Step 2.

Step 6. The estimated stability region and the LQ controller are obtained by (38) and (39).

Remark 13: From Lemma 3 and Remark 2, the condition for Algorithm SYNTHESIS to be applicable is

$$\begin{aligned} &-A^{\mathrm{T}}P - PA - (\gamma_1+\gamma_2)P^2 - \frac{1}{\gamma_1}\alpha^2 I \\ &-\frac{1}{\gamma_2}\beta^2 PBR^{-2}B^{\mathrm{T}}P + 2PBR^{-1}B^{\mathrm{T}}P > 0 \end{aligned} \qquad (44)$$

Since (44) is not an LMI of P, R, γ_1 and γ_2, we denote $\hat{P} = P^{-1}$, $\hat{R} = R^{-1}$, then (44) becomes

$$\begin{aligned} &-\hat{P}A^{\mathrm{T}} - A\hat{P} - (\gamma_1+\gamma_2)I - \frac{1}{\gamma_1}\alpha^2\hat{P}^2 \\ &-\frac{1}{\gamma_2}\beta^2 B\hat{R}^2B^{\mathrm{T}} + 2B\hat{R}B^{\mathrm{T}} > 0 \end{aligned}$$

which is LMI

$$\begin{bmatrix} -\hat{P}A^{\mathrm{T}} - A\hat{P} - (\gamma_1+\gamma_2)I + 2B\hat{R}B^{\mathrm{T}} & \alpha\hat{P} & \beta B\hat{R} \\ \alpha\hat{P} & \gamma_1 I & 0 \\ \beta\hat{R}B^{\mathrm{T}} & 0 & \gamma_2 I \end{bmatrix} > 0 \qquad (45)$$

From (40), a choice of the initial X_0 to guarantee the feasibility of the LMI problem in Step 3 is as follows. Solve LMI (45) to obtain $\hat{P}$ and $\hat{R}$, then let

$$X_0 = \tfrac{1}{4}\lambda_{\min}^2(\Gamma_{\hat{P}^{-1}}\hat{R}^{-1})\hat{P}$$

Remark 14: Since $X \ge X_0$ implies $\Gamma_{X^{-1}} \ge \Gamma_{X_0^{-1}}$, (42) and (43) imply (36). Hence X, $\tilde{W}$, $\tilde{R}$, $\gamma_{1\lambda}, \gamma_{2\lambda}$ obtained in Step 3 satisfies the MIs in Theorem 3. Moreover, once the LMI problem in Step 3 is feasible for an X_0, it is also feasible for the next iterate defined in Step 5. Furthermore, the positive sequence $\{\det(X)\}$ obtained in Algorithm SYNTHESIS is non-decreasing. If it is bounded, then $\{\det(X)\}$ will converge and the algorithm will stop. If $\det(X) \to +\infty$, then the volume of the stability region can be made as large as one wants and we can stop the algorithm when a required value of $\det(X)$ is obtained.

Remark 15: When $\Delta A = \Delta B = 0$. The largest estimated stability region is (38) where $X \in \mathbb{R}^{n\times n} > 0$ is the solution of MI problem

$$\max \det(X) \text{subject to}$$

$$-XA^{\mathrm{T}} - AX + B\Gamma_{X^{-1}}B^{\mathrm{T}} > 0 \qquad (46)$$

Moreover, a corresponding LQ controller can be chosen as

$$u_{\max} = -\Gamma_{X^{-1}}B^{\mathrm{T}}X^{-1}x \qquad (47)$$

4.2. *System with structured uncertainties*

When ΔA and ΔB have the structures (29) with (30) and $E_2^{\mathrm{T}} E_2$ diagonal, the following corollary can be obtained.

Corollary 3: *Suppose* $X \in \mathbb{R}^{n\times n} > 0$, $\tilde{W} = diag[\tilde{w}_1, \ldots, \tilde{w}_m] \in \mathbb{R}^{m\times m} > 0$, $\tilde{R} = diag[\tilde{r}_1, \ldots, \tilde{r}_m] \in \mathbb{R}^{m\times m} > 0$, $\gamma_{1\lambda}, \gamma_{2\lambda} \in \mathbb{R}$ *are the solutions of the MI problem*

$$\max \det(X) \quad \textit{subject to } (35), (36), (37) \textit{ and}$$

$$\begin{bmatrix} -XA^{\mathrm{T}} - AX - \gamma_{1\lambda} D_1 D_1^{\mathrm{T}} - \gamma_{2\lambda} D_2 D_2^{\mathrm{T}} + B\tilde{W}B^{\mathrm{T}} & XE_1^{\mathrm{T}} & B\tilde{R}E_2^{\mathrm{T}} \\ XE_1 & \gamma_{1\lambda} I & 0 \\ E_2 \tilde{R} B^{\mathrm{T}} & 0 & \gamma_{2\lambda} I \end{bmatrix} > 0 \tag{48}$$

then the largest estimated stability region is obtained as (38) *and the LQ controller is given by* (39).

Remark 16: The only difference between Corollary 3 and Theorem 3 is the LMIs (48) and (34). Hence Algorithm SYNTHESIS can also be used to design the controller (substitute (34) with (48) in Step 3).

5. Examples

5.1. *Example 1*

Consider $\mathcal{S}$ with saturation function (2) described by Kim and Bien (1994)

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3.0 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta_1 = 5, \quad \Delta_2 = 2$$

5.1.1. *Estimation of stability region*. To compare our estimation results with existing ones, we keep $R = \mathrm{diag}[2, 1]$ fixed and choose three different typical Qs

$$Q_1 = \mathrm{diag}[1, 10], \quad Q_2 = \mathrm{diag}[1, 1], \quad Q_3 = \mathrm{diag}[10, 1] \tag{49}$$

Three Qs are chosen to give a better reflection of the results due to the choice of Q in the estimation process. The above three Qs lead to three LQ controllers and therefore three closed-loop systems.

Case of unstructured uncertainties: As in Kim and Bien (1994), we suppose $\alpha = \beta = 0.1$ in (7). Using Procedure ESTIMATION, we obtain the estimated stability regions

$$\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2913x_1^2 + 0.027x_1x_2 + 1.3583x_2^2 < 4087.2\}$$

$$\Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2909x_1^2 - 0.0038x_1x_2 + 0.1623x_2^2 < 3997.9\}$$

and

$$\Omega_3 = \{(x_1, x_2) \in \mathbb{R}^2 : 0.9024x_1^2 - 0.0103x_1x_2 + 0.1624x_2^2 < 10699\}$$

respectively. Graphical comparisons between our estimation and the best estimation using the three methods presented in Kim and Bien (1994) are given in figures 2–4. It can be seen that our estimation is better than that in Kim and Bien (1994).

Case of structured uncertainties: Now consider the case when ΔA and ΔB have the structures in (29) with

$$D_1 = \frac{1}{20}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad E_1 = [1 \quad 1], \quad D_2 = \frac{1}{10} I, \quad E_2 = I \tag{50}$$

It is easy to see that this case is more special than the norm-bounded uncertainties $\|\Delta A\| \le 0.1$ and $\|\Delta B\| \le 0.1$. However, it is more general than that in Henrion and Tarbouriech (1999) and Henrion *et al.* (1999), where

$$[\Delta A \quad \Delta B] = DF(t)[E_1 \quad E_2]$$

with

$$D = \frac{1}{20}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad E_1 = E_2 = [1 \quad 1]$$

In this case, for the three Qs in (49), we obtain the estimated stability regions

$$\hat{\Omega}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2913x_1^2 + 0.027x_1x_2 + 1.3583x_2^2 < 6636.1\}$$

$$\hat{\Omega}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2909x_1^2 - 0.0038x_1x_2 + 0.1623x_2^2 < 7547.5\}$$

and

$$\hat{\Omega}_3 = \{(x_1, x_2) \in \mathbb{R}^2 : 0.9024x_1^2 - 0.0103x_1x_2 + 0.1624x_2^2 < 20325\}$$

respectively. Graphical comparisons between our estimation and that obtained by using the method in Henrion and Tarbouriech (1999) are also given in figures 2–4. It can be seen that the volume of our estimation is

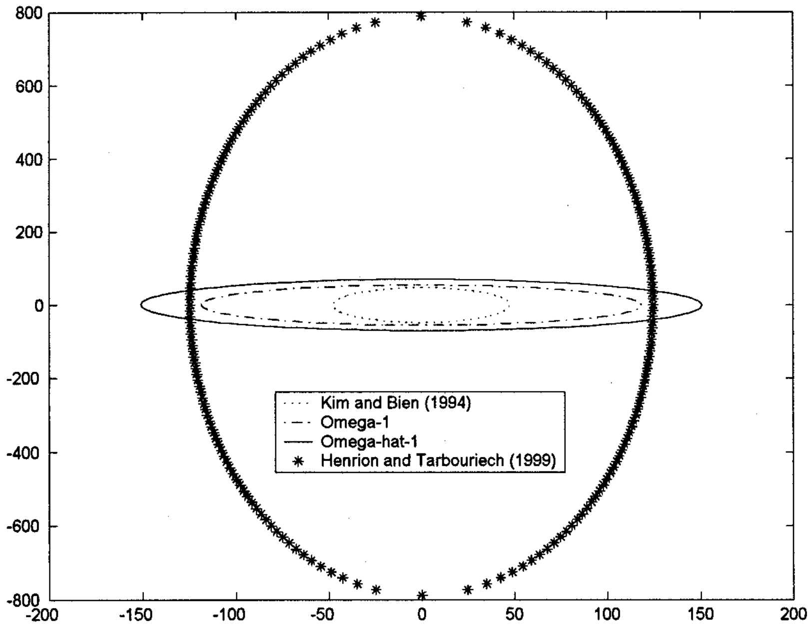

Figure 2. Case of uncertainties: $Q_1 = \mathrm{diag}\,[1, 10]$.

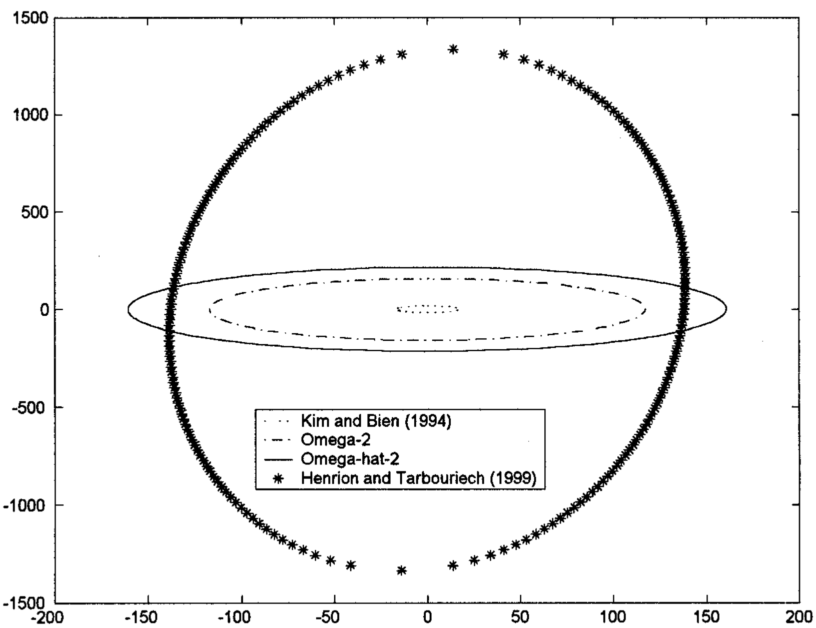

Figure 3. Case of uncertainties: $Q_2 = \mathrm{diag}\,[1, 1]$.

smaller than that of Henrion and Tarbouriech (1999), the reasons may be due to those pointed out in Remarks 10 and 11.

Case of no uncertainties: When $\Delta A = \Delta B = 0$, we obtain the three estimated stability regions

$$\Omega_{01} = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2913x_1^2 + 0.027x_1x_2 + 1.3583x_2^2 < 16361\}$$

$$\Omega_{02} = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2909x_1^2 - 0.0038x_1x_2 + 0.1623x_2^2 < 18393\}$$

$$\Omega_{03} = \{(x_1, x_2) \in \mathbb{R}^2 : 0.9024x_1^2 - 0.0103x_1x_2 + 0.1624x_2^2 < 54921\}$$

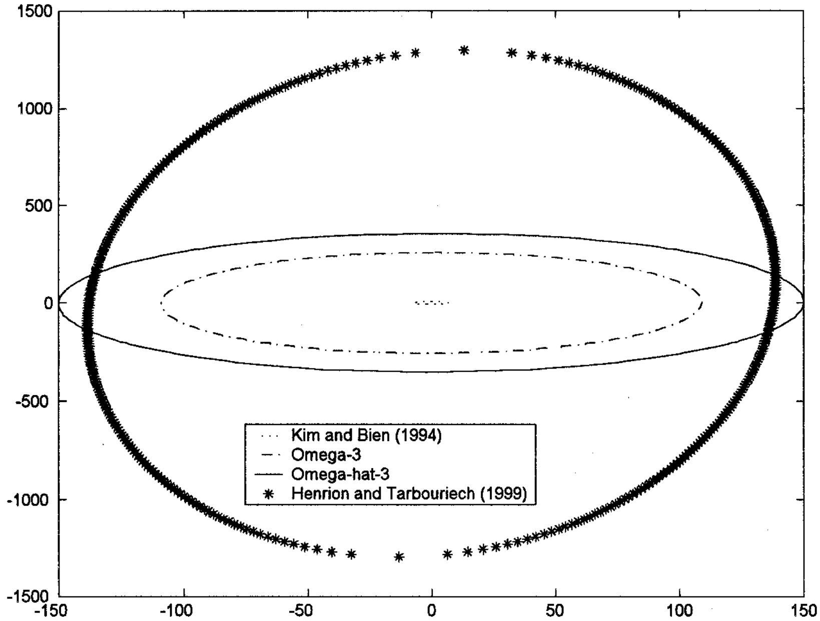

Figure 4. Case of uncertainties: $Q_3 = \text{diag}\,[10, 1]$.

The estimation obtained by the methods in Kim and Bien (1994), Gomes da Silva Jr and Tarbouriech (1999 a)

$$\left(\text{where } G = \begin{bmatrix} I \\ -I \end{bmatrix}\right)$$

and Pittet *et al.* (1997) (circle criterion method) are also given in figures 5–7. It can be seen from figures 6 and 7 that our estimation is better than that in Kim and Bien (1994) and Gomes da Silva Jr and Tarbouriech (1999 a) when $Q_2 = \text{diag}[1,\ 1]$ and $Q_3 = \text{diag}[10,\ 1]$.

5.1.2. *Controller design*

Case of unstructured uncertainties: When $\alpha = \beta = 0.1$ in (7), using Theorem 3 and Algorithm SYNTHESIS, we obtain the largest estimated stability region

$$\Omega_{\max} = \{(x_1, x_2) \in \mathbb{R}^2 : 1.1373x_1^2 - 0.0722x_1x_2 + 0.0076x_2^2 < 10\,000\} \tag{51}$$

with the LQ controller

$$u_{\max} = \begin{bmatrix} -0.0533 & 0.0017 \\ 0.0083 & -0.0017 \end{bmatrix} x$$

Case of structured uncertainties: When ΔA and ΔB have the structures in (29) with (50), using Corollary 3 and Algorithm SYNTHESIS, we obtain the largest estimated stability region

$$\hat{\Omega}_{\max} = \{(x_1, x_2) \in \mathbb{R}^2 : 0.7755x_1^2 - 0.0444x_1x_2 + 0.0021x_2^2 < 10000\}$$

with the LQ controller

$$\hat{u}_{\max} = \begin{bmatrix} -0.0440 & 0.0013 \\ 0.0098 & -0.0009 \end{bmatrix} x$$

The initial state set obtained in Henrion *et al.* (1999) is

$$\mathcal{D}_0 = \{(x_1, x_2) \in \mathbb{R}^2 : 2.905x_1^2 - 0.159x_1x_2 + 0.005x_2^2 < 10\,000\}$$

The regions $\hat{\Omega}_{\max}$ and $\mathcal{D}_0$ are compared in figure 8. The volume ratio between $\hat{\Omega}_{\max}$ and $\mathcal{D}_0$ is 2.7242. ($\hat{\Omega}_{\max}$ is also compared with $\Omega_{\max}$ in figure 8.)

Case of no uncertainties: When $\Delta A = \Delta B = 0$, we obtain the largest estimated stability region

$$\Omega_{0\,\max} = \{(x_1, x_2) \in \mathbb{R}^2 : 0.1491x_1^2 - 0.0096x_1x_2 + 0.00016x_2^2 < 10\,000\}$$

with the LQ controller

$$u_{0\,\max} = \begin{bmatrix} -0.0386 & 0.0012 \\ 0.0152 & -0.0005 \end{bmatrix} x$$

5.2. *Example 2*

Consider $\mathcal{S}$ with saturation function (2) described by Henrion *et al.* (1999)

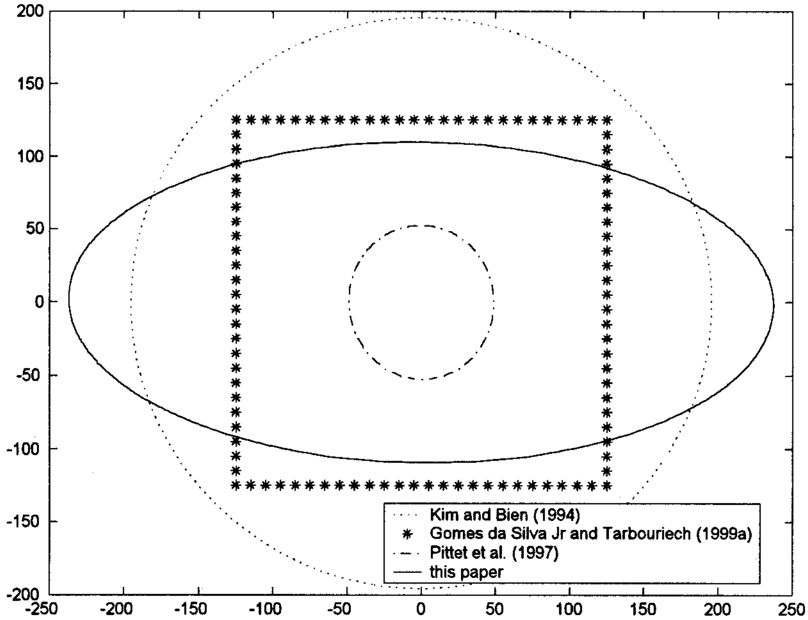

Figure 5. Case of no uncertainties: $Q_1 = \mathrm{diag}[1, 10]$.

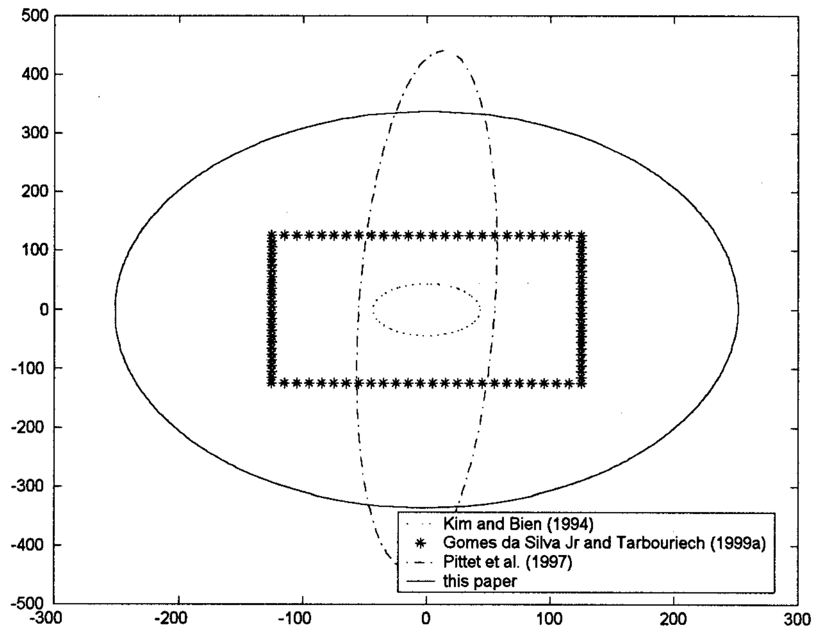

Figure 6. Case of no uncertainties: $Q_2 = [1, 1]$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.75 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \Delta_1 = \Delta_2 = 15$$

As in Henrion *et al.* (1999), suppose ΔA and ΔB have the structures in (29) with

$$D_1 = D_2 = [0, 3, 0, 1]^{\mathrm{T}}, \quad E_1 = [1, 1, 0, 0.333], \quad E_2 = 0$$

Using Corollary 3 and Algorithm SYNTHESIS, we obtain the largest estimated stability region

$$\Omega_{\max} = \{x \in \mathbb{R}^4 : x^{\mathrm{T}} P x < 1\}$$

where

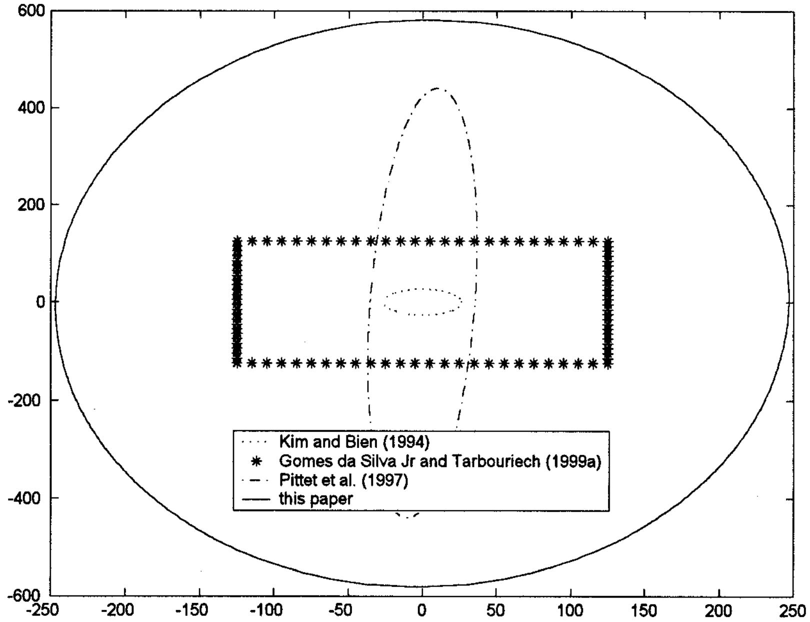

Figure 7. Case of no uncertainties: $Q_3 = \mathrm{diag}[10, 1]$.

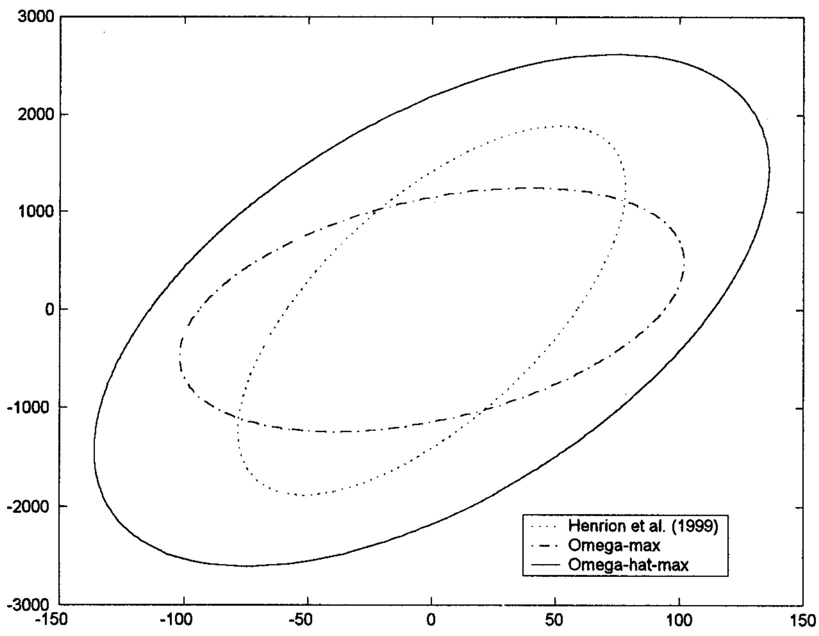

Figure 8. Case of uncertainties: maximization.

$$P = 10^{-8}\begin{bmatrix} 9342900 & 4617100 & -5.5601 & 4126200 \\ 4617100 & 3069100 & -2.0403 & 1880500 \\ -5.5601 & -2.0403 & 1.0093 & -0.1263 \\ 4126200 & 1880500 & -0.1263 & 1854900 \end{bmatrix}$$

and the LQ controller

$$u_{\max} = 10^4\begin{bmatrix} -1.1325 & -0.7528 & 0.0000 & -0.4612 \\ -1.0122 & -0.4613 & 0.0000 & -0.4550 \end{bmatrix}x$$

The initial state set obtained in Henrion *et al.* (1999) is

$$\mathcal{D}_0 = \left\{x \in \mathbb{R}^4 : x^{\mathrm{T}} S x < 1\right\}$$

where

$$S = 10^{-3}\begin{bmatrix} 420.7 & 254.8 & -0.396 & 174.9 \\ 254.8 & 191.9 & -1.594 & 95.14 \\ -0.396 & -1.594 & 0.838 & 2.129 \\ 174.9 & 95.14 & 2.129 & 80.47 \end{bmatrix}$$

$\mathcal{D}_0$ is contained in $\Omega_{\max}$ because $S - P > 0$. The volume ratio between $\Omega_{\max}$ and $\mathcal{D}_0$ is 3933.5.

6. Conclusion

In this paper, we first give a number of new methods to estimate the stability region for uncertain linear systems with saturating LQ control laws. Then an ILMI algorithm is presented to design the controller such that the above estimated stability region is the largest.

It should be noted that no assumptions on the dimension of the system and the number of open-loop unstable poles are needed in our methods. Moreover, the number of LMIs (MIs) will not increase when the dimension of the system increases, which is different from the polytopic representation method used in Henrion and Tarbouriech (1999) and Henrion *et al.* (1999). Thus our method is likely to be more useful to high dimensional systems with more open-loop unstable poles. The examples have demonstrated that our design method can obtain a much larger estimated stability region than that of Henrion *et al.* (1999), especially for higher dimensional systems.

A limitation of the proposed methods is that it can only be applied to LQ controller. However, less conservative results can be obtained by exploiting the special structure of the controller which is otherwise impossible for a generic controller. Since LQ controller plays a significant part in control theory and applications, the estimation results obtained in this paper provide a useful *a posteriori* analysis on a given LQ controller under saturation, while the synthesis results provide an enhanced approach for LQ design.

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