



H_∞ model reduction for linear time-delay systems: continuous-time case

SHENGYUAN XU[†], JAMES LAM[‡], SHOUDONG HUANG[‡] and CHENGWU YANG[§]

This paper deals with the problem of H_∞ model reduction for linear continuous time-delay systems. For a given delay system, the problem we address is the construction of a reduced-order model such that the associated model error satisfies a prescribed H_∞ norm bound constraint. Two alternative methods for obtaining reduced-order models are presented. Sufficient conditions for the existence of desired reduced-order models are proposed in terms of linear matrix inequalities (LMIs) and a coupling non-convex rank constraint set. Conditions based on strict LMIs are obtained for the zeroth-order H_∞ approximation problem. When these conditions are satisfied, an explicit parametrization of the desired reduced-order models is also presented. All these results are extended to time-delay systems with parameter uncertainties. Finally, an illustrative example is provided to demonstrate the effectiveness of the proposed approach.

1. Introduction

It is well known that mathematical modelling of physical systems often results in high-order models. In practical applications, it is desirable to replace these high-order models with reduced ones with respect to some given criterion. This has motivated the study of the model reduction problem with various approaches such as the balanced truncation method and the optimal Hankel norm approximation method (Desai and Pal 1984, Glover 1984, Sreeram and Athoklis 1991, Chiu 1996, Yan and Lam 1999). When parameter uncertainties appear in these high-order models, some results on robust model reduction have also been reported (Beck *et al.* 1996, Haddad and Kapila 1997).

Model reduction of delay systems has received much attention in the past decades, especially for systems with input/output delays. In the Laplace domain, the classic approaches are related to the continued-fraction or Padé approximation of the delay term (Marshak *et al.* 1974, Lam 1993), and the partial fraction method (Zwart *et al.* 1988). The Hankel operator approach and convergence properties were studied in Glover *et al.* (1990). On the other hand, parametric optimization approaches based on the H_2 norm have been considered by many researchers (see Zhang and Lam 1999 and references therein).

Recently, the so-called H_∞ model reduction problem has received much attention. The problem involves approximating a stable system with n states by another stable system with $\bar{n} < n$ states such that the associated model error satisfies a prescribed H_∞ norm bound con-

straint (Kavranoğlu 1992, 1993, Kavranoğlu and Bettayeb 1993, Grigoriadis 1995). The H_∞ model reduction problem was addressed in Kavranoğlu and Bettayeb (1993) by converting the problem into a Hankel norm model reduction through an imbedding process. In Kavranoğlu (1992, 1993) the zeroth-order H_∞ model reduction problem concerning the choice of a constant appropriately to approximate a given system in the sense of H_∞ norm bound constraints was addressed and a characterization of the solutions to the problem was presented by using a similar approach. On the other hand, in Grigoriadis (1995) a linear matrix inequality method was proposed and necessary and sufficient conditions were obtained for the existence of solutions to the continuous-time and discrete-time H_∞ model reduction problems. It should be pointed out that all these H_∞ model reduction results are derived in the context of continuous- and discrete-time systems without delays and parameter uncertainties. Up to date, it appears that no effort has been made to extend these results to the case of time-delay systems, whether with or without parameter uncertainties.

In this paper, we consider the H_∞ model reduction problem for linear continuous time-delay systems. For a given stable delay system, the objective is to construct a stable reduced-order model such that the associated model reduction error meets a prescribed H_∞ norm bound constraint. Two alternative reduced-order models are considered, where one involves state time-delays, while the other does not. Based on the results of H_∞ analysis for time-delay systems, sufficient conditions for the existence of solutions to the problem are obtained in terms of LMIs and a coupling non-convex rank constraint, and an explicit parametrization of the desired reduced-order models is given. Particularly strict LMI conditions are derived for the zeroth-order H_∞ approximation problem, and a parametrization of the solutions are also presented. Furthermore, in the case when parameter uncertainties appear in the system matrices,

Received 19 September 2000. Revised 26 January 2001.

[†] Author for correspondence. This author was with the 810 Division, School of Power Engineering, Nanjing University of Science and Technology, Nanjing, 210094, P.R. China.

[‡] Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong.

[§] 810 Division, School of Power Engineering, Nanjing University of Science and Technology, Nanjing, 210094, P.R. China.

sufficient conditions for the robust H_∞ model reduction problem are proposed, and a parametrization of the desired reduced-order models is given.

Throughout this paper, for symmetric matrices X and Y , the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimension. The notation M^T represents the transpose of the matrix M . For a given stable continuous-time transfer function matrix $G(s)$, its H_∞ norm is given by $\|G(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(j\omega)]$, where σ_{\max} represents the maximum singular value of a matrix. For a matrix $M \in \mathbb{R}^{n \times m}$ with rank r , the orthogonal complement M^\perp is defined as a (possibly non-unique) $(n - r) \times n$ matrix with rank $n - r$, such that $M^\perp M = 0$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. H_∞ model reduction for time-delay systems with no uncertainty

Consider a stable linear continuous time-delay system Σ described by

$$\dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) + C_d x(t-d) + Du(t) \quad (2)$$

$$x(t) = \phi(t), \quad t \in [-d, 0] \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output, A, A_d, B, C and D are known real constant matrices. $d > 0$ is the constant time-delay of the system, and $\phi(t)$ is the continuous initial value function on $[-d, 0]$. The transfer function from $u(t)$ to $y(t)$ is given by

$$G(s) = (C + C_d e^{-sd})(sI - A - A_d e^{-sd})^{-1}B + D \quad (4)$$

Then the H_∞ model reduction problem we address first can be formulated as finding a stable time-delay system $\hat{\Sigma}$ given by

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{A}_d\hat{x}(t-d) + \hat{B}u(t) \quad (5)$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{C}_d\hat{x}(t-d) + \hat{D}u(t) \quad (6)$$

$$\hat{x}(t) = \psi(t), \quad t \in [-d, 0] \quad (7)$$

where $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$, $\hat{y}(t) \in \mathbb{R}^p$, $\psi(t)$ is a continuous value function on $[-d, 0]$, and $\hat{n} < n$, such that

$$\|G(s) - \hat{G}_d(s)\|_\infty \leq \gamma \quad (8)$$

where

$$\hat{G}_d(s) = (\hat{C} + \hat{C}_d e^{-sd})(sI - \hat{A} - \hat{A}_d e^{-sd})^{-1}\hat{B} + \hat{D} \quad (9)$$

and γ is a prescribed positive scalar. As in other model reduction formulations, the initial condition $\phi(t)$ and $\psi(t)$ are assumed to be zero for $t \in [-d, 0]$.

If $\hat{n} = 0$, then the reduced system (5) and (7) is only a constant \hat{D} , in this case, the model reduction problem reduces to the so-called zeroth-order H_∞ approximation problem (Kavranoğlu 1992, 1993).

Before proceeding further, we give the following lemmas which will be used in the proof of our main results.

Lemma 1: Consider the time-delay system (1)–(3). If there exist matrices $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} PA + A^T P + C^T C + Q & C^T C_d + P A_d & C^T D + P B \\ C_d^T C + A_d^T P & C_d^T C_d - Q & C_d^T D \\ D^T C + B^T P & D^T C_d & D^T D - \gamma^2 I \end{bmatrix} < 0 \quad (10)$$

then $\|G(s)\|_\infty \leq \gamma$.

Proof: The proof can be carried out by following similar lines as in the proof of Theorem 1 in Lee *et al.* (1994) and Theorem 3.2 in Mahmoud and Zribi (1999), and thus is omitted. \square

Lemma 2 (Gahinet and Apkarian 1994, Iwasaki and Skelton 1994): Given a symmetric matrix Ξ and two matrices Γ and Π , consider the problem of finding some matrix Θ such that

$$\Xi + \Gamma \Theta \Pi + (\Gamma \Theta \Pi)^T < 0 \quad (11)$$

Then (11) is solvable for Θ if and only if

$$\Gamma^\perp \Xi \Gamma^{\perp T} < 0, \quad \Pi^{\perp T} \Xi \Pi^{\perp T} < 0$$

We are now in a position to give a condition for the solvability of the continuous-time H_∞ model reduction problem for system (1)–(3).

Theorem 1: If there exist matrices $X > 0, Y > 0$ and $Z > 0$ satisfying

$$\begin{bmatrix} A^T X + X A + Z & X A_d & C^T \\ A_d^T X & -Z & C_d^T \\ C & C_d & -\gamma I \end{bmatrix} < 0 \quad (12)$$

$$\begin{bmatrix} A^T Y + Y A + Z + Y - X & Y A_d & Y B \\ A_d^T Y & -Z + X - Y & 0 \\ B^T Y & 0 & -\gamma I \end{bmatrix} < 0 \quad (13)$$

$$Y - X \leq 0 \quad (14)$$

and

$$\text{rank}(Y - X) \leq \hat{n} \tag{15}$$

then there exists a stable time-delay system $\hat{\Sigma}$ to solve the continuous-time H_∞ model reduction problem. In this case, a desired reduced system corresponding to a feasible solution (X, Y, Z) to (12)–(15) is given by

$$\begin{aligned} & \begin{bmatrix} \hat{D} & \hat{C} & \hat{C}_d \\ \hat{B} & \hat{A} & \hat{A}_d \end{bmatrix} \\ & = -W^{-1}\Psi^T A \Phi^T (\Phi A \Phi^T)^{-1} + W^{-1}S^{1/2}L(\Phi A \Phi^T)^{-1/2} \\ & S = W - \Psi^T [A - A \Phi^T (\Phi A \Phi^T)^{-1} \Phi A] \Psi \\ & A = (\Psi W^{-1} \Psi^T - \Omega)^{-1} \\ & \Omega = \begin{bmatrix} A^T X + XA + Z & A^T X_{12} - X_{12} & XA_d & 0 & XB & C^T \\ X_{12}^T A - X_{12}^T & X_{22} & X_{12}^T A_d & 0 & X_{12}^T B & 0 \\ A_d^T X & A_d^T X_{12} & -Z & X_{12} & 0 & C_d^T \\ 0 & 0 & X_{12}^T & -X_{22} & 0 & 0 \\ B^T X & B^T X_{12} & 0 & 0 & -\gamma I & D^T \\ C & 0 & C_d & 0 & D & -\gamma I \end{bmatrix} \\ & \Psi = \begin{bmatrix} 0 & X_{12} \\ 0 & X_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -I & 0 \end{bmatrix} \\ & \Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} \end{aligned}$$

where L is any matrix satisfying $\|L\| < 1$, and $X_{12} \in \mathbb{R}^{n \times \hat{n}}$, $X_{22} \in \mathbb{R}^{\hat{n} \times \hat{n}}$, $X_{22} > 0$ and $W > 0$ satisfying

$$A > 0, \quad Y - X = -X_{12} X_{22}^{-1} X_{12}^T \leq 0$$

Proof: From (1)–(3), (5) and (7), the state-space representation of the error system $\tilde{G} = G - \hat{G}_d$ can be written as

$$\dot{\hat{x}}(t) = \tilde{A}\hat{x}(t) + \tilde{A}_d\hat{x}(t-d) + \tilde{B}u(t) \tag{16}$$

$$\hat{y}(t) = \tilde{C}\hat{x}(t) + \tilde{C}_d\hat{x}(t-d) + \tilde{D}u(t) \tag{17}$$

where

$$\hat{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \quad \hat{y}(t) = y(t) - \hat{y}(t) \tag{18}$$

$$\left. \begin{aligned} \tilde{A} &= \bar{A} + \bar{F}\bar{G}\bar{H}, & \tilde{A}_d &= \bar{A}_d + \bar{F}\bar{G}\bar{K} \\ \tilde{B} &= \bar{B} + \bar{F}\bar{G}\bar{N} \end{aligned} \right\} \tag{19}$$

$$\left. \begin{aligned} \tilde{C} &= \bar{C} + \bar{S}\bar{G}\bar{H}, & \tilde{C}_d &= \bar{C}_d + \bar{S}\bar{G}\bar{K} \\ \tilde{D} &= \bar{D} + \bar{S}\bar{G}\bar{N} \end{aligned} \right\} \tag{20}$$

$$\left. \begin{aligned} \bar{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, & \bar{A}_d &= \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, & \bar{F} &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\ \bar{G} &= \begin{bmatrix} \hat{D} & \hat{C} & \hat{C}_d \\ \hat{B} & \hat{A} & \hat{A}_d \end{bmatrix}, & \bar{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix} \end{aligned} \right\} \tag{21}$$

$$\bar{H} = \begin{bmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

$$\left. \begin{aligned} \bar{C} &= [C \ 0], & \bar{C}_d &= [C_d \ 0] \\ \bar{S} &= [-I \ 0], & \bar{D} &= D \end{aligned} \right\} \tag{23}$$

By considering (14) and (15), we can deduce that there exist matrices $Y_{12} \in \mathbb{R}^{n \times \hat{n}}$, $Y_{22} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ and $Y_{22} > 0$ such that

$$Y - X = -Y_{12} Y_{22}^{-1} Y_{12}^T \leq 0 \tag{24}$$

Let

$$P = \begin{bmatrix} X & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Z & -Y_{12} \\ -Y_{12}^T & Y_{22} \end{bmatrix} \tag{25}$$

Then, by some algebraic manipulations we have

$$P^{-1} = \begin{bmatrix} Y^{-1} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \tag{26}$$

where

$$Z_{12} = -Y^{-1} Y_{12} Y_{22}^{-1}, \quad Z_{22} = (Y_{22} - Y_{12}^T X^{-1} Y_{12})^{-1} \tag{27}$$

From (13) and (24), it can be deduced that $Q > 0$. Next we shall show that under the conditions of the theorem there exists a matrix \bar{G} such that

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} + Q & P \tilde{A}_d & P \tilde{B} & \tilde{C}^T \\ \tilde{A}_d^T P & -Q & 0 & \tilde{C}_d^T \\ \tilde{B}^T P & 0 & -\gamma I & \tilde{D}^T \\ \tilde{C} & \tilde{C}_d & \tilde{D} & -\gamma I \end{bmatrix} < 0 \tag{28}$$

To this end, we use the expressions (19)–(23) and rewrite (28) as

$$\Omega_1 + \Psi_1 \bar{G} \Phi_1 + (\Psi_1 \bar{G} \Phi_1)^T < 0 \tag{29}$$

where

$$\Omega_1 = \begin{bmatrix} \bar{A}^T P + P\bar{A} + Q & P\bar{A}_d & P\bar{B} & \bar{C}^T \\ \bar{A}_d^T P & -Q & 0 & \bar{C}_d^T \\ \bar{B}^T P & 0 & -\gamma I & \bar{D}^T \\ \bar{C} & \bar{C}_d & \bar{D} & -\gamma I \end{bmatrix}$$

$$\Psi_1 = \begin{bmatrix} P\bar{F} \\ 0 \\ 0 \\ \bar{S} \end{bmatrix}, \quad \Phi_1 = [\bar{H} \quad \bar{K} \quad \bar{N} \quad 0]$$

By Lemma 2 it is easy to see that a necessary and sufficient condition for LMI (29) to have a solution \bar{G} is

$$\Psi_1^\perp \Omega_1 \Psi_1^{\perp T} < 0, \quad \Phi_1^{\perp T} \Omega_1 \Phi_1^{\perp} < 0 \quad (30)$$

Noting that

$$\Psi_1^\perp = \begin{bmatrix} [I \ 0] & [0 \ 0] & 0 & 0 \\ [0 \ 0] & [I \ 0] & 0 & 0 \\ [0 \ 0] & [0 \ I] & 0 & 0 \\ [0 \ 0] & [0 \ 0] & I & 0 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

$$\Phi_1^{\perp T} = \begin{bmatrix} [I \ 0] & [0 \ 0] & 0 & 0 \\ [0 \ 0] & [I \ 0] & 0 & 0 \\ [0 \ 0] & [0 \ 0] & 0 & I \end{bmatrix}$$

we can show that (see below).

Considering (12), (13) and (24), it follows that the matrix P defined in (25) satisfies (30). Therefore, there exists a matrix \bar{G} such that (28) holds. Now, by Lemma 1 and Schur complements, it can be easily shown that $\|G(s) - \bar{G}(s)\|_\infty \leq \gamma$ holds. That is, the H_∞ model reduction problem is solvable. Furthermore, when (12)–(15) are satisfied, the parametrization of all reduced models satisfying the LMI (29) can be obtained by using

the results in Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994). This completes the proof. \square

Remark 1: Theorem 1 provides a sufficient condition for the existence of solutions to the H_∞ model reduction problem for continuous time-delay systems. Obviously, inequalities (12)–(15) are non-convex though the constraints (12)–(14) are convex. Fortunately, an efficient numerical algorithm based on alternating projections to solve (12)–(15) can be available in Grigoriadis (1995). Moreover, as in Grigoriadis (1995) a bisection approach can be used to seek for the minimum H_∞ norm bound γ in order to solve the optimal H_∞ model reduction problem for continuous time-delay systems.

In Theorem 1 the desired reduced order system involves time-delays. However, in some applications it is desirable to replace the time-delay system (1)–(3) with a reduced system with no time-delays, and the associated model error also achieves a prescribed H_∞ norm condition. In view of this, we now consider the H_∞ model reduction problem with a reduced model with no time-delays. More specifically, we formulate the H_∞ model reduction problem for the time-delay system (1)–(3) as finding a stable \hat{n} th order system $\hat{\Sigma}'$ given by

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \quad (31)$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t) \quad (32)$$

where $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$, $\hat{y}(t) \in \mathbb{R}^p$, such that

$$\|G(s) - \hat{G}(s)\|_\infty \leq \gamma \quad (33)$$

where $\gamma > 0$ is a prescribed scalar, $G(s)$ is defined in (4), and

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \quad (34)$$

Theorem 2: *If there exist matrices $X > 0$, $Y > 0$ and $Z > 0$ satisfying*

$$\Psi_1^\perp \Omega_1 \Psi_1^{\perp T} = \begin{bmatrix} Y^{-1}A^T + AY^{-1} + Y^{-1}ZY^{-1} - Y^{-1}Y_{12}Y_{22}^{-1}Y_{12}^T Y^{-1} & A_d & 0 & B \\ & A_d^T & -Z & Y_{12} & 0 \\ & 0 & Y_{12}^T & -Y_{22} & 0 \\ & B^T & 0 & 0 & -\gamma I \end{bmatrix}$$

$$\Phi_1^{\perp T} \Omega_1 \Phi_1^{\perp} = \begin{bmatrix} A^T X + XA + Z & XA_d & C^T \\ A_d^T X & -Z & C_d^T \\ C & C_d & -\gamma I \end{bmatrix}$$

$$\begin{bmatrix} A^T X + XA + Z & XA_d & C^T \\ A_d^T X & -Z & C_d^T \\ C & C_d & -\gamma I \end{bmatrix} < 0 \quad (35)$$

$$\begin{bmatrix} A^T Y + YA + Z & YA_d & YB \\ A_d^T Y & -Z & 0 \\ B^T Y & 0 & -\gamma I \end{bmatrix} < 0 \quad (36)$$

$$Y - X \leq 0 \quad (37)$$

and

$$\text{rank}(Y - X) \leq \hat{n} \quad (38)$$

then there exists a stable \hat{n} th order system $\hat{\Sigma}'$ to solve the continuous-time H_∞ model reduction problem. In this case, a desired \hat{n} th order system corresponding to a feasible solution (X, Y, Z) to (35)–(38) is given by

$$\begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} = -W^{-1} \Psi^T A \Phi^T (\Phi A \Phi^T)^{-1} + W^{-1} S^{1/2} L (\Phi A \Phi^T)^{-1/2}$$

$$S = W - \Psi^T [A - A \Phi^T (\Phi A \Phi^T)^{-1} \Phi A] \Psi$$

$$A = (\Psi W^{-1} \Psi^T - \Omega)^{-1}$$

$$\Omega = \begin{bmatrix} A^T X + XA + Z & A^T X_{12} & XA_d & XB & C^T \\ X_{12}^T A & 0 & X_{12}^T A_d & X_{12}^T B & 0 \\ A_d^T X & A_d^T X_{12} & -Z & 0 & C_d^T \\ B^T X & B^T X_{12} & 0 & -\gamma I & D^T \\ C & 0 & C_d & D & -\gamma I \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 0 & X_{12} \\ 0 & X_{22} \\ 0 & 0 \\ 0 & 0 \\ -I & 0 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix}$$

where L is any matrix satisfying $\|L\| < 1$, and $X_{12} \in \mathbb{R}^{n \times \hat{n}}$, $X_{22} \in \mathbb{R}^{\hat{n} \times \hat{n}}$, $X_{22} > 0$ and $W > 0$ satisfying

$$A > 0, \quad Y - X = -X_{12} X_{22}^{-1} X_{12}^T \leq 0$$

Proof: From (1)–(3), (31) and (32), the state-space representation of the error system $\tilde{G} = G - \hat{G}$ can be written as

$$\dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{A}_d \tilde{x}(t-d) + \tilde{B} u(t) \quad (39)$$

$$\tilde{y}(t) = \tilde{C} \tilde{x}(t) + \tilde{C}_d \tilde{x}(t-d) + \tilde{D} u(t) \quad (40)$$

where

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \quad \tilde{y}(t) = y(t) - \hat{y}(t) \quad (41)$$

$$\left. \begin{aligned} \tilde{A} &= \bar{A} + \bar{F} \bar{G} \bar{H}, & \tilde{B} &= \bar{B} + \bar{F} \bar{G} \bar{N} \\ \tilde{C} &= \bar{C} + \bar{S} \bar{G} \bar{H}, & \tilde{D} &= \bar{D} + \bar{S} \bar{G} \bar{N} \end{aligned} \right\} \quad (42)$$

$$\left. \begin{aligned} \bar{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, & \bar{A}_d &= \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{F} = \bar{H} &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, & \bar{G} &= \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} \end{aligned} \right\} \quad (43)$$

$$\left. \begin{aligned} \bar{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix} \\ \bar{N} &= \begin{bmatrix} I \\ 0 \end{bmatrix}, & \bar{C} &= [C \quad 0], & \bar{C}_d &= [C_d \quad 0] \\ \bar{S} &= [-I \quad 0], & \bar{D} &= D \end{aligned} \right\} \quad (44)$$

Along the same line as in the proof of Theorem 1, we can show that under the conditions of the theorem there exist a matrix \bar{G} such that

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} + \bar{Q} & P \tilde{A}_d & P \tilde{B} & \tilde{C}^T \\ \tilde{A}_d^T P & -Z & 0 & C_d^T \\ \tilde{B}^T P & 0 & -\gamma I & \tilde{D}^T \\ \tilde{C} & C_d & \tilde{D} & -\gamma I \end{bmatrix} < 0 \quad (45)$$

where

$$\bar{Q} = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix} \quad (46)$$

It is easy to see that (45) implies that there exists a scalar $\delta > 0$ such that

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} + \bar{Q} & P \tilde{A}_d & P \tilde{B} & \tilde{C}^T \\ \tilde{A}_d^T P & -Z & 0 & C_d^T \\ \tilde{B}^T P & 0 & -\gamma I & \tilde{D}^T \\ \tilde{C} & C_d & \tilde{D} & -\gamma I \end{bmatrix} < 0 \quad (47)$$

where

$$Q = \begin{bmatrix} Z & 0 \\ 0 & \delta I \end{bmatrix}$$

Then, from (47) it can be verified that

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} + Q & P\tilde{A}_d & P\tilde{B} & \tilde{C}^T \\ \tilde{A}_d^T P & -Q & 0 & \tilde{C}_d^T \\ \tilde{B}^T P & 0 & -\gamma I & \tilde{D}^T \\ \tilde{C} & \tilde{C}_d & \tilde{D} & -\gamma I \end{bmatrix} < 0$$

Finally, using Lemma 1 and Schur complements, the desired results follow immediately. \square

Remark 2: In the case when $A_d = 0$ and $C_d = 0$, that is, the time-delay system (1)–(3) reduces to a state-space system with no time-delays. It is easy to show that Theorem 1 and Theorem 2 coincide with Theorem 1 in Grigoriadis (1995). In view of this, Theorem 1 and Theorem 2 can be viewed as extensions of existing results on H_∞ model reduction to time-delay systems.

From Theorems 1 and 2 we can obtain a strict LMI condition for the solvability of the zeroth-order H_∞ approximation.

Corollary 1: *If there exist matrices $X > 0$, $Z > 0$ satisfying the following LMIs*

$$\begin{bmatrix} A^T X + XA + Z & XA_d & C^T \\ A_d^T X & -Z & C_d^T \\ C & C_d & -\gamma I \end{bmatrix} < 0 \quad (48)$$

$$\begin{bmatrix} A^T X + XA + Z & XA_d & XB \\ A_d^T X & -Z & 0 \\ B^T X & 0 & -\gamma I \end{bmatrix} < 0 \quad (49)$$

then there exists a constant \hat{D} to solve the zeroth-order H_∞ approximation problem. In this case, a desired zeroth-order H_∞ solution \hat{D} corresponding to a feasible matrix pair (X, Z) is given by

$$\begin{aligned} \hat{D} &= -W^{-1} \Psi^T \Lambda \Phi^T (\Phi \Lambda \Phi^T)^{-1} \\ &\quad + W^{-1} S^{1/2} L (\Phi \Lambda \Phi^T)^{-1/2} \\ S &= W - \Psi^T [\Lambda - \Lambda \Phi^T (\Phi \Lambda \Phi^T)^{-1} \Phi \Lambda] \Psi \\ \Lambda &= (\Psi W^{-1} \Psi^T - \Omega)^{-1} \end{aligned}$$

$$\Omega = \begin{bmatrix} A^T X + XA + Z & XA_d & XB & C^T \\ A_d^T X & -Z & 0 & C_d^T \\ B^T X & 0 & -\gamma I & D^T \\ C & C_d & D & -\gamma I \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -I \end{bmatrix}$$

$$\Phi = [0 \quad 0 \quad I \quad 0]$$

where L is any matrix satisfying $\|L\| < 1$ and $W > 0$ satisfying $\Lambda > 0$.

3. H_∞ model reduction for time-delay systems with uncertainties

In this section, we shall consider the H_∞ model reduction problem for uncertain continuous time-delay systems.

Consider the uncertain continuous time-delay system

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) \\ &\quad + (A_d + \Delta A_d)x(t-d) + Bu(t) \end{aligned} \quad (50)$$

$$\begin{aligned} y(t) &= (C + \Delta C)x(t) \\ &\quad + (C_d + \Delta C_d)x(t-d) + Du(t) \end{aligned} \quad (51)$$

$$x(t_0 + \theta) = \phi(\theta), \quad \forall \theta \in [-d, 0] \quad (52)$$

where ΔA , ΔA_d , ΔC , and ΔC_d are parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix} \Delta A & \Delta A_d \\ \Delta C & \Delta C_d \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F [N_1 \quad N_2] \quad (53)$$

where M_1 , M_2 , N_1 and N_2 are constant matrices, and $F \in \mathbb{R}^{i \times j}$ is the uncertain matrix satisfying

$$F^T F \leq I \quad (54)$$

The parameter uncertainties ΔA , ΔA_d , ΔC and ΔC_d are said to be admissible if both (53) and (54) hold.

The H_∞ model reduction problem we address first in this section is to design a stable time-delay system Σ such that

$$\|G_u(s) - \hat{G}_d(s)\|_\infty \leq \gamma \quad (55)$$

holds for all admissible uncertainties, where

$$G_u(s) = [C + \Delta C + (C_d + \Delta C_d) e^{-sd}] \\ \times [sI - A - \Delta A - (A_d + \Delta A_d) e^{-sd}]^{-1} B + D$$

$$\left. \begin{aligned} V &= (\epsilon_1^{-1} + \epsilon_2^{-1}) M_1 M_1^T \\ J &= (\epsilon_3^{-1} + \epsilon_4^{-1}) M_2 M_2^T \end{aligned} \right\} \quad (61)$$

and $\hat{G}_d(s)$ are defined in (9).

We first introduce the following lemma which will be used in the proof of our main results in this section.

Lemma (Li and De Souza 1997): *Let D , E and F be real matrices of appropriate dimensions with F satisfying $F^T F \leq I$. Then, for any scalar $\epsilon > 0$*

$$DFE + (DFE)^T \leq \epsilon^{-1} DD^T + \epsilon E^T E$$

The following result provides a sufficient condition for the solution of the H_∞ model reduction problem for system (50)–(52).

Theorem 3: *If there exist matrices $X > 0$, $Y > 0$, $Z > 0$ and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\epsilon_4 > 0$ satisfying (equations 56–58 see below) and*

$$\text{rank}(Y - X) \leq \hat{n} \quad (59)$$

where

$$U = (\epsilon_1 + \epsilon_3) N_1^T N_1 \quad (60)$$

then there exists a reduced system Σ such that (55) holds for all admissible uncertainties. Furthermore, in this case, the desired reduced systems corresponding to a feasible solution $(X, Y, Q, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ to (56)–(59) are parameterized as (see equations, top of next page), where

L is any matrix satisfying $\|L\| < 1$, and $X_{12} \in \mathbb{R}^{n \times \hat{n}}$, $X_{22} \in \mathbb{R}^{\hat{n} \times \hat{n}}$, $X_{22} > 0$ and $W > 0$ satisfying

$$A > 0, \quad Y - X = -X_{12} X_{22}^{-1} X_{12}^T \leq 0 \quad (62)$$

Proof: Using the notation in (18)–(23), we obtain the state-space representation of the error system $\tilde{G}_u = G_u - \hat{G}_d$ as

$$\dot{\tilde{x}}(t) = \tilde{A}_u \tilde{x}(t) + \tilde{A}_{du} \tilde{x}(t-d) + \tilde{B}u(t) \quad (63)$$

$$\tilde{y}(t) = \tilde{C}_u \tilde{x}(t) + \tilde{C}_{du} \tilde{x}(t-d) + \tilde{D}u(t) \quad (64)$$

where

$$\left. \begin{aligned} \tilde{A}_u &= \tilde{A} + \Delta \tilde{A}, & \tilde{A}_{du} &= \tilde{A}_d + \Delta \tilde{A}_d \\ \tilde{C}_u &= \tilde{C} + \Delta \tilde{C}, & \tilde{C}_{du} &= \tilde{C}_d + \Delta \tilde{C}_d \end{aligned} \right\} \quad (65)$$

$$\left. \begin{aligned} \Delta \tilde{A} &= \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, & \Delta \tilde{A}_d &= \begin{bmatrix} \Delta A_d & 0 \\ 0 & 0 \end{bmatrix} \\ \Delta \tilde{C} &= [\Delta C \quad 0], & \Delta \tilde{C}_d &= [\Delta C_d \quad 0] \end{aligned} \right\} \quad (66)$$

From (58) and (59) we can define matrices $P > 0$ and $Q > 0$ as in (25). Following a similar line as in the proof of Theorem 1, it can be shown that under the conditions of the theorem there exists a matrix \tilde{G} such that

$$\Omega_{1u} + \Psi_{1u} \tilde{G} \Phi_{1u} + (\Psi_{1u} \tilde{G} \Phi_{1u})^T < 0 \quad (67)$$

where

$$\left[\begin{array}{ccccccc} A^T X + XA + Z + U & XA_d & C^T & XM_1 & XM_1 & 0 & 0 \\ A_d^T X & -Z + (\epsilon_2 + \epsilon_4) N_2^T N_2 & C_d^T & 0 & 0 & 0 & 0 \\ C & C_d & -\gamma I & 0 & 0 & M_2 & M_2 \\ M_1^T X & 0 & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ M_1^T X & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ 0 & 0 & M_2^T & 0 & 0 & -\epsilon_3 I & 0 \\ 0 & 0 & M_2^T & 0 & 0 & 0 & -\epsilon_4 I \end{array} \right] < 0 \quad (56)$$

$$\left[\begin{array}{ccccc} A^T Y + YA + Z + Y - X + U & YA_d & YB & YM_1 & YM_1 \\ A_d^T Y & -Z + X - Y + (\epsilon_2 + \epsilon_4) N_2^T N_2 & 0 & 0 & 0 \\ B^T Y & 0 & -\gamma I & 0 & 0 \\ M_1^T Y & 0 & 0 & -\epsilon_1 I & 0 \\ M_1^T Y & 0 & 0 & 0 & -\epsilon_2 I \end{array} \right] < 0 \quad (57)$$

$$Y - X \leq 0 \quad (58)$$

$$\begin{bmatrix} \hat{D} & \hat{C} & \hat{C}_d \\ \hat{B} & \hat{A} & \hat{A}_d \end{bmatrix} = -W^{-1}\Psi^T\Lambda\Phi^T(\Phi\Lambda\Phi^T)^{-1}$$

$$+ W^{-1}S^{1/2}L(\Phi\Lambda\Phi^T)^{-1/2}$$

$$S = W - \Psi^T[\Lambda - \Lambda\Phi^T(\Phi\Lambda\Phi^T)^{-1}\Phi\Lambda]\Psi$$

$$\Lambda = (\Psi W^{-1}\Psi^T - \Omega)^{-1}$$

$$\Omega = \begin{bmatrix} A^T X + XA + Z + U + XVX & (A^T + XV - I)X_{12} & XA_d & 0 & XB & C^T \\ X_{12}^T(A + VX - I) & X_{22} + X_{12}^T VX_{12} & X_{12}^T A_d & 0 & X_{12}^T B & 0 \\ A_d^T X & A_d^T X_{12} & -Z + (\epsilon_2 + \epsilon_4)N_2^T N_2 & X_{12} & 0 & C_d^T \\ 0 & 0 & X_{12}^T & -X_{22} & 0 & 0 \\ B^T X & B^T X_{12} & 0 & 0 & -\gamma I & D^T \\ C & 0 & C_d & 0 & D & -\gamma I + J \end{bmatrix}$$

$$\Psi^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -I \\ X_{12}^T & X_{22} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}$$

$$\Omega_{1u} = \begin{bmatrix} \bar{A}^T P + P\bar{A} + Q + P\tilde{V}P + \bar{\Xi}_1 & P\bar{A}_d & P\bar{B} & \bar{C}^T \\ \bar{A}_d^T P & -Q + \bar{\Xi}_2 & 0 & \bar{C}_d^T \\ \bar{B}^T P & 0 & -\gamma I & \bar{D}^T \\ \bar{C} & \bar{C}_d & \bar{D} & -\gamma I + J \end{bmatrix}$$

$$\Psi_{1u} = \begin{bmatrix} P\bar{F} \\ 0 \\ 0 \\ \bar{S} \end{bmatrix}, \quad \Phi_{1u} = [\bar{H} \quad \bar{K} \quad \bar{N} \quad 0]$$

and

$$\tilde{V} = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\Xi}_1 = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{\Xi}_2 = \begin{bmatrix} (\epsilon_2 + \epsilon_4)N_2^T N_2 & 0 \\ 0 & 0 \end{bmatrix}$$

That is,

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + Q + P\tilde{V}P + \bar{\Xi}_1 & P\bar{A}_d & P\bar{B} & \bar{C}^T \\ \bar{A}_d^T P & -Q & 0 & \bar{C}_d^T \\ \bar{B}^T P & 0 & -\gamma I & \bar{D}^T \\ \bar{C} & \bar{C}_d & \bar{D} & -\gamma I \end{bmatrix} < 0 \tag{68}$$

On the other hand, using Lemma 3 we have

$$\Delta\tilde{A}^T P + P\Delta\tilde{A} \leq P\Sigma_1 P + \Pi_1$$

$$\begin{bmatrix} 0 & P\Delta\tilde{A}_d \\ \Delta\tilde{A}_d^T P & 0 \end{bmatrix} \leq \begin{bmatrix} P\Sigma_2 P & 0 \\ 0 & \Pi_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & \Delta\tilde{C}^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta\tilde{C} & 0 & 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \Pi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_3^{-1} M_2 M_2^T \end{bmatrix}$$

$$\begin{bmatrix} -Q & 0 & \Delta\tilde{C}_d^T \\ 0 & 0 & 0 \\ \Delta\tilde{C}_d & 0 & 0 \end{bmatrix} \leq \begin{bmatrix} -Q + \Pi_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_4^{-1} M_2 M_2^T \end{bmatrix}$$

where

$$\left. \begin{aligned} \Sigma_1 &= \begin{bmatrix} \epsilon_1^{-1} M_1 M_1^T & 0 \\ 0 & 0 \end{bmatrix}, & \Sigma_2 &= \begin{bmatrix} \epsilon_2^{-1} M_1 M_1^T & 0 \\ 0 & 0 \end{bmatrix}, \\ \Pi_1 &= \begin{bmatrix} \epsilon_1 N_1^T N_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \right\} \tag{69}$$

$$\left. \begin{aligned} \Pi_2 &= \begin{bmatrix} \epsilon_2 N_2^T N_2 & 0 \\ 0 & 0 \end{bmatrix}, & \Pi_3 &= \begin{bmatrix} \epsilon_3 N_1^T N_1 & 0 \\ 0 & 0 \end{bmatrix}, \\ \Pi_4 &= \begin{bmatrix} \epsilon_4 N_2^T N_2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \right\} \quad (70)$$

Then, it follows from (68) that

$$\begin{bmatrix} (\tilde{A} + \Delta\tilde{A})^T P + P(\tilde{A} + \Delta\tilde{A}) + Q & P(\tilde{A}_d + \Delta\tilde{A}_d) & P\tilde{B} & (\tilde{C} + \Delta\tilde{C})^T \\ (\tilde{A}_d + \Delta\tilde{A}_d)^T P & -Q & 0 & (\tilde{C}_d + \Delta\tilde{C}_d)^T \\ \tilde{B}^T P & 0 & -\gamma I & \tilde{D}^T \\ \tilde{C} + \Delta\tilde{C} & \tilde{C}_d + \Delta\tilde{C}_d & \tilde{D} & -\gamma I \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} \tilde{A}_u^T P + P\tilde{A}_u + Q & P\tilde{A}_{du} & P\tilde{B} & \tilde{C}_u^T \\ \tilde{A}_{du}^T P & -Q & 0 & \tilde{C}_{du}^T \\ \tilde{B}^T P & 0 & -\gamma I & \tilde{D}^T \\ \tilde{C}_u & \tilde{C}_{du} & \tilde{D} & -\gamma I \end{bmatrix} < 0$$

Hence, from Lemma 1 we can deduce that the H_∞ model reduction problem for system (50)–(52) is solvable. This completes the proof. \square

Theorem 1 gives a sufficient condition for the solvability of the H_∞ model reduction problem for uncertain system (50)–(52) by using a reduced model with time-delays. When the desired reduced model involves no time-delays, we have the following result.

Theorem 4: *If there exist matrices $X > 0$, $Y > 0$, $Z > 0$ and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\epsilon_4 > 0$ satisfying (equations 71–73, see below) and*

$$\text{rank}(Y - X) \leq \hat{n} \quad (74)$$

where U is defined in (60), then there exists a reduced system $\tilde{\Sigma}'$ such that (55) holds for all admissible uncertainties. In this case, a desired reduced system corresponding to a feasible solution $(X, Y, Q, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ to (71)–(74) is given by equations (top of next page), where V and J are defined in (61), L is any matrix satisfying $\|L\| < 1$, and $X_{12} \in \mathbb{R}^{n \times \hat{n}}$, $X_{22} \in \mathbb{R}^{\hat{n} \times \hat{n}}$, $X_{22} > 0$ and $W > 0$ satisfy-

$$A > 0, \quad Y - X = -X_{12} X_{22}^{-1} X_{12}^T \leq 0 \quad (75)$$

Proof: Along a similar line as in the proof of Theorem 2 and Theorem 3, the desired result follows. \square

From Theorem 3 and Theorem 4, we can also obtain a strict LMI condition for the solvability of the zeroth-order H_∞ approximation for system (50)–(52).

Corollary 2: *If there exist matrices $X > 0$, $Z > 0$ and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\epsilon_4 > 0$ satisfying the LMIs (see equations (76)–(77) following) where*

$$U = (\epsilon_1 + \epsilon_3) N_1^T N_1 \quad (78)$$

then there exists a constant \hat{D} to solve the zeroth-order H_∞ approximation problem. In this case, a desired zeroth-order H_∞ solution \hat{D} corresponding to a feasible solution $(X, Y, Q, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ to (56)–(59) is given by

$$\begin{bmatrix} A^T X + XA + Z + U & XA_d & C^T & XM_1 & XM_1 & 0 & 0 \\ A_d^T X & -Z + (\epsilon_2 + \epsilon_4) N_2^T N_2 & C_d^T & 0 & 0 & 0 & 0 \\ C & C_d & -\gamma I & 0 & 0 & M_2 & M_2 \\ M_1^T X & 0 & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ M_1^T X & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ 0 & 0 & M_2^T & 0 & 0 & -\epsilon_3 I & 0 \\ 0 & 0 & M_2^T & 0 & 0 & 0 & -\epsilon_4 I \end{bmatrix} < 0 \quad (71)$$

$$\begin{bmatrix} A^T Y + YA + Z + U & YA_d & YB & YM_1 & YM_1 \\ A_d^T Y & -Z + (\epsilon_2 + \epsilon_4) N_2^T N_2 & 0 & 0 & 0 \\ B^T Y & 0 & -\gamma I & 0 & 0 \\ M_1^T Y & 0 & 0 & -\epsilon_1 I & 0 \\ M_1^T Y & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0 \quad (72)$$

$$Y - X \leq 0 \quad (73)$$

$$\begin{bmatrix} \hat{D} & \hat{C} & \hat{C}_d \\ \hat{B} & \hat{A} & \hat{A}_d \end{bmatrix} = -W^{-1}\Psi^T\Lambda\Phi^T(\Phi\Lambda\Phi^T)^{-1} + W^{-1}S^{1/2}L(\Phi\Lambda\Phi^T)^{-1/2}$$

$$S = W - \Psi^T[\Lambda - \Lambda\Phi^T(\Phi\Lambda\Phi^T)^{-1}\Phi\Lambda]\Psi, \quad \Lambda = (\Psi W^{-1}\Psi^T - \Omega)^{-1}$$

$$\Omega = \begin{bmatrix} A^T X + XA + Z + U + XVX & (A^T + XV)X_{12} & XA_d & XB & C^T \\ X_{12}^T(A + VX) & X_{12}^T VX_{12} & X_{12}^T A_d & X_{12}^T B & 0 \\ A_d^T X & A_d^T X_{12} & -Z + (\epsilon_2 + \epsilon_4)N_2^T N_2 & 0 & C_d^T \\ B^T X & B^T X_{12} & 0 & -\gamma I & D^T \\ C & 0 & C_d & D & -\gamma I + J \end{bmatrix}$$

$$\Psi^T = \begin{bmatrix} 0 & 0 & 0 & 0 & -I \\ X_{12}^T & X_{22} & 0 & 0 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{D} = -W^{-1}\Psi^T\Lambda\Phi^T(\Phi\Lambda\Phi^T)^{-1} + W^{-1}S^{1/2}L(\Phi\Lambda\Phi^T)^{-1/2}$$

$$S = W - \Psi^T[\Lambda - \Lambda\Phi^T(\Phi\Lambda\Phi^T)^{-1}\Phi\Lambda]\Psi$$

$$\Lambda = (\Psi W^{-1}\Psi^T - \Omega)^{-1}$$

where L is any matrix satisfying $\|L\| < 1$ and $W > 0$ satisfying $\Lambda > 0$.

Remark 3: Along the lines of argument similar to Theorem 3 and Theorem 4, the corresponding results can be extended to the case when parameter uncertainties also appear in the matrices B and D .

4. An illustrative example

In this section, we present an illustrative example to demonstrate the applicability of the proposed approach.

Consider an uncertain linear continuous time-delay system (50)–(52) with

$$\Omega = \begin{bmatrix} A^T X + XA + Z + U + XVX & XA_d & XB & C^T \\ A_d^T X & -Z + (\epsilon_2 + \epsilon_4)N_2^T N_2 & 0 & C_d^T \\ B^T X & 0 & -\gamma I & D^T \\ C & C_d & D & -\gamma I + J \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -I \end{bmatrix}, \quad \Phi = [0 \ 0 \ I \ 0]$$

$$\begin{bmatrix} A^T X + XA + Z + U & XA_d & C^T & XM_1 & XM_1 & 0 & 0 \\ A_d^T X & -Z + (\epsilon_2 + \epsilon_4)N_2^T N_2 & C_d^T & 0 & 0 & 0 & 0 \\ C & C_d & -\gamma I & 0 & 0 & M_2 & M_2 \\ M_1^T X & 0 & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ M_1^T X & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ 0 & 0 & M_2^T & 0 & 0 & -\epsilon_3 I & 0 \\ 0 & 0 & M_2^T & 0 & 0 & 0 & -\epsilon_4 I \end{bmatrix} < 0 \quad (76)$$

$$\begin{bmatrix} A^T X + XA + Z + U & XA_d & XB & XM_1 & XM_1 \\ A_d^T X & -Z + (\epsilon_2 + \epsilon_4)N_2^T N_2 & 0 & 0 & 0 \\ B^T X & 0 & -\gamma I & 0 & 0 \\ M_1^T X & 0 & 0 & -\epsilon_1 I & 0 \\ M_1^T X & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0 \quad (77)$$

$$A = \begin{bmatrix} -11 & 1 & 1 & 1 & 0 & 0 \\ 1 & -10 & -2 & 1 & 0.3 & 0 \\ -1 & 0 & -7 & 1 & 0 & 1 \\ 0 & 1 & 0 & -4 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 & 0 & -10 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 1 & 0.5 & 0 & 1 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0.1 & 0 \\ 0.5 & -1 & 1 & 0.2 & 0 & 0.2 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ -0.5 & 0 & 0.2 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$C_d = \begin{bmatrix} 0 & 0.3 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \Delta A & \Delta A_d \\ \Delta C & \Delta C_d \end{bmatrix}$$

$$= \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F [N_1 \mid N_2]$$

$$= \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0 \\ 0.1 \end{bmatrix} \sin \alpha [0.1 \ 0 \ 0.1 \ 0.1 \mid 0 \ 0.1 \ 0.1 \ 0.1 \ 0 \ 0 \ 0.1 \ 0.2]$$

The time delay d is supposed to be 1.6. In this example, we set $\gamma = 2.6$. It is required to find a stable first order

system (5)–(7) such that (55) is satisfied. By solving (56)–(59) we obtain

$$X = \begin{bmatrix} 1.4084 & 0.0742 & -0.1058 & -0.0580 & -0.0733 & 0.0175 \\ 0.0742 & 1.2999 & -0.3212 & 0.0209 & -0.0773 & -0.0315 \\ -0.1058 & -0.3212 & 1.7865 & -0.1524 & -0.2251 & 0.0282 \\ -0.0580 & 0.0209 & -0.1524 & 2.3653 & -0.1838 & 0.0913 \\ -0.0733 & -0.0773 & -0.2251 & -0.1838 & 2.5822 & -0.2297 \\ 0.0175 & -0.0315 & 0.0282 & 0.0913 & -0.2297 & 1.3262 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.9084 & 0.0742 & -0.1058 & -0.0580 & -0.0733 & 0.0175 \\ 0.0742 & 1.2999 & -0.3212 & 0.0209 & -0.0773 & -0.0315 \\ -0.1058 & -0.3212 & 1.7865 & -0.1524 & -0.2251 & 0.0282 \\ -0.0580 & 0.0209 & -0.1524 & 2.3653 & -0.1838 & 0.0913 \\ -0.0733 & -0.0773 & -0.2251 & -0.1838 & 2.5822 & -0.2297 \\ 0.0175 & -0.0315 & 0.0282 & 0.0913 & -0.2297 & 1.3262 \end{bmatrix}$$

$$Z = \begin{bmatrix} 12.3601 & -0.9352 & -0.5492 & -1.1855 & -1.1163 & -0.9659 \\ -0.9352 & 11.7715 & -1.8297 & -2.0521 & -1.4671 & -0.2537 \\ -0.5492 & -1.8297 & 10.6115 & -1.5637 & -1.1561 & -0.2747 \\ -1.1855 & -2.0521 & -1.5637 & 7.8749 & -0.1273 & -0.0376 \\ -1.1163 & -1.4671 & -1.1561 & -0.1273 & 6.4881 & -1.9497 \\ -0.9659 & -0.2537 & -0.2747 & -0.0376 & -1.9497 & 12.0658 \end{bmatrix}$$

$$\epsilon_1 = 11.1817, \quad \epsilon_2 = 10.9434, \quad \epsilon_3 = 10.9453, \quad \epsilon_4 = 10.6013$$

Then, from Theorem 3 the H_∞ model reduction problem is solvable. It is easy to show that

$$Y - X = \begin{bmatrix} -0.5000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(Y - X) = 1$$

Therefore, we can choose

$$X_{12}^T = [1 \ 0 \ 0 \ 0 \ 0 \ 0], \quad X_{22} = 2$$

$$L = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 \\ 0 & 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0.99 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}$$

and hence

$$\Psi^T = \left[\begin{array}{cccccc|c|cccccc|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Phi = \left[\begin{array}{cccccc|c|cccccc|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

A desired reduced model is given by

$$\begin{bmatrix} \hat{D} & \hat{C} & \hat{C}_d \\ \hat{B} & \hat{A} & \hat{A}_d \end{bmatrix} = \left[\begin{array}{ccc|cc} 2.1984 & 0.0781 & 1.3964 & -1.7080 & 0.0945 \\ -0.4339 & 2.8748 & 1.3809 & -0.6285 & 0.1088 \\ \hline -0.9056 & -0.6267 & 0.4650 & -14.8258 & -0.2586 \end{array} \right]$$

That is

$$\begin{aligned} \dot{\hat{x}}(t) &= -14.8258\hat{x}(t) - 0.2586\hat{x}(t-d) \\ &\quad + [-0.9056 \quad -0.6267 \quad 0.4650]u(t) \\ \hat{y}(t) &= \begin{bmatrix} -1.7080 \\ -0.6285 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0.0945 \\ 0.1088 \end{bmatrix} \hat{x}(t-d) \\ &\quad + \begin{bmatrix} 2.1984 & 0.0781 & 1.3964 \\ -0.4339 & 2.8748 & 1.3809 \end{bmatrix} u(t) \end{aligned}$$

which is stable and meets the prescribed H_∞ constraint. The output trajectories of the original system and the reduced order system are compared in figures 1 and 2 (the control input was chosen as $u_i(t) = \exp(-t)$, $i = 1, 2, 3$, and the parameter uncertainty $\alpha = 0.5$).

5. Conclusions

In this paper, we have studied the problem of H_∞ model reduction for linear continuous time-delay systems. Both systems with and without parameter uncertainties are considered. Sufficient conditions for the existence of solutions to the problem have been obtained in terms of LMIs and a non-convex coupling constraint, and a parametrization of the desired reduced model corresponding to a feasible matrix solution has been given. Two alternative methods for designing reduced-order models have been proposed. An illustrative example has been given to demonstrate the effectiveness of the proposed approach.

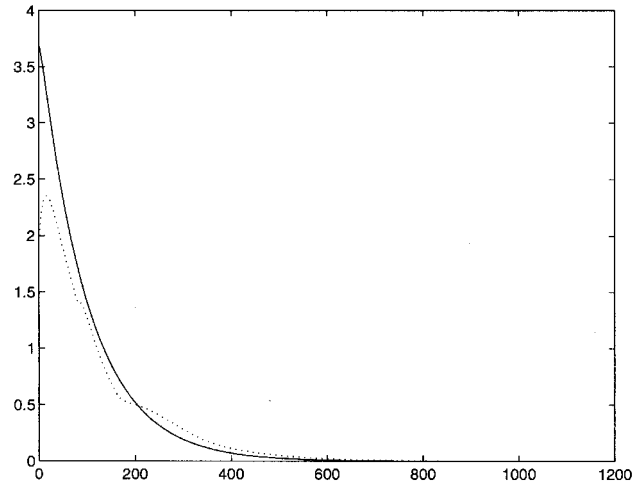


Figure 1. $y_1(t)$ (\cdots) and $\hat{y}_1(t)$ ($—$).

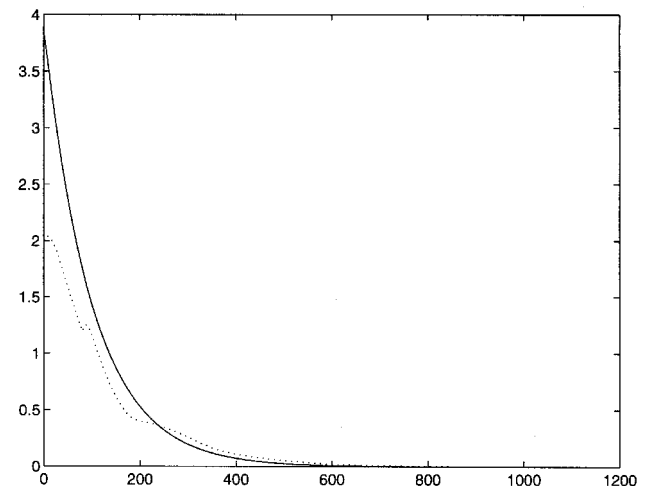


Figure 2. $y_2(t)$ (\cdots) and $\hat{y}_2(t)$ ($—$).

Acknowledgements

This work is partially supported by HKU CRCG 10202590/19870.

References

- BECK, C. L., DOYLE, J., and GLOVER, K., 1996, Model reduction of multidimensional and uncertain systems. *IEEE Transactions on Automatic Control*, **41**, 1466–1477.
- CHIU, T., 1996, Model reduction by the low-frequency approximation balancing method for unstable systems. *IEEE Transactions on Automatic Control*, **41**, 995–997.
- DESAI, U. B., and PAL, D., 1984, A transformation approach to stochastic model reduction. *IEEE Transactions on Automatic Control*, **29**, 1097–1100.
- GAHINET, P., and APKARIAN, P., 1994, A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, **4**, 421–448.
- GLOVER, K., 1984, All optimal Hankel–Norm approximations of linear multivariable systems and their L_∞ error bounds. *International Journal of Control*, **39**, 1115–1193.
- GLOVER, K., LAM, J., and PARTINGTON, J. R., 1990, Rational approximation of a class of infinite dimensional systems with optimal rate of convergence. I: Singular values of Hankel operator. *Mathematics of Control, Signals and Systems*, **3**, 325–344.
- GRIGORIADIS, K. M., 1995, Optimal H_∞ model reduction via linear matrix inequalities: continuous- and discrete-time cases. *Systems and Control Letters*, **26**, 321–333.
- HADDAD, W. M., and KAPILA, V., 1997, Robust reduced-order modelling for state-space systems via parameter-dependent bounding functions. *IEEE Transactions on Automatic Control*, **42**, 248–253.
- IWASAKI, T., and SKELTON, R. E., 1994, All controllers for the general H_∞ control problems: LMI existence conditions and state space formulas. *Automatica*, **30**, 1307–1317.
- KAVRANOĞLU, D., 1992, H_∞ norm approximation of systems by constant matrices and related results. *Proceedings of 31st IEEE Conference on Decision and Control*, Tucson, Arizona, USA, pp. 3271–3275.
- KAVRANOĞLU, D., 1993, Zeroth order H_∞ norm approximation of multivariable systems. *Numerical Functional Analysis and Optimization*, **14**, 89–101.
- KAVRANOĞLU, D., and BETTAYEB, M., 1993, Characterization of the solution to the optimal H_∞ model reduction problem. *Systems and Control Letters*, **20**, 99–107.
- LAM, J., 1993, Model reduction of delay systems using Padé approximants. *International Journal of Control*, **57**, 377–391.
- LEE, J. H., KIM, S. W., and KWON, W. H., 1994, Memoryless H_∞ controllers for state delayed systems. *IEEE Transactions on Automatic Control*, **39**, 159–162.
- LI, X., and DE SOUZA, C. E., 1997, Criteria for robust stability and stabilization of uncertain linear systems with state-delay. *Automatica*, **33**, 1657–1662.
- MAHMOUD, M. S., and ZRIBI, M., 1999, H_∞ controllers for time-delay systems using linear matrix inequalities. *Journal of Optimization Theory and Applications*, **100**, 89–121.
- MARSHAK, A. H., JOHNSON, D. E., and JOHNSON, J. R., 1974, A Bessel rational filter. *IEEE Transactions on Circuits Systems*, **21**, 797–799.
- SREERAM, V., and AGATHOKLIS, P., 1991, Model reduction of linear discrete systems via weighted impulse response Gramians. *International Journal of Control*, **53**, 129–144.
- YAN, W. Y., and LAM, J., 1999, An approximate approach to H_2 optimal model reduction. *IEEE Transactions on Automatic Control*, **44**, 1341–1358.
- ZHANG, L., and LAM, J., 1999, Optimal weighted L_2 model reduction of delay systems. *International Journal of Control*, **72**, 39–48.
- ZWART, H. J., CURTAIN, R. F., PARTINGTON, J. R., and GLOVER, K., 1988, Partial fraction expansions for delay systems. *Systems and Control Letters*, **10**, 235–244.