

On the Number of Local Minima to the Point Feature Based SLAM Problem

Shoudong Huang, Heng Wang, Udo Frese, and Gamini Dissanayake

Abstract—Map joining is an efficient strategy for solving feature based SLAM problems. This paper demonstrates that joining of two local maps, formulated as a nonlinear least squares problem has at most two local minima, when the associated uncertainties can be described using spherical covariance matrices. Necessary and sufficient condition for the existence of two minima is derived and it is shown that more than one minimum exists only when the quality of the local maps used for map joining is extremely poor. The analysis explains to some extent why a number of optimization based SLAM algorithms proposed in the recent literature that rely on local search strategies are successful in converging to the globally optimal solution from poor initial conditions, particularly when covariance matrices are spherical. It also demonstrates that the map joining problem has special properties that may be exploited to reliably obtain globally optimal solutions to the SLAM problem.

I. INTRODUCTION

When SLAM problem is formulated as a nonlinear least squares problem, the dimension of the problem is very high because all feature positions and robot poses are present as variables. It can be expected that such high dimensional nonlinear optimization problem have a huge number of local minima and in general local search strategies are unlikely to be successful unless a very good initial guess is available. However, recent research shows that some methods based on local search can sometimes provide surprisingly good solutions to SLAM without being trapped into a local minimum.

For pose-only SLAM problems, the work proposed in [1] surprised many SLAM researchers where stochastic gradient descent (SGD) is used to solve the optimization problem by dealing with each constraint individually and the algorithm can converge to the correct solution with poor initial values. Recently, a more efficient SLAM algorithm, tree-based network optimizer (TORO), was proposed in [2] where a tree structure is used on top of the SGD approach. Surprisingly, very large scale problems can be solved efficiently without the need of good initial values, especially when the covariance matrices of the relative poses are close to spherical [3].

Our initial investigation [4] into point feature based SLAM, formulated as a nonlinear least squares problem has

also highlighted some interesting behavior. A simple Gauss-Newton algorithm can sometimes converge to the global optimal solution from random initial values, when used with the popular Victoria Park dataset [5]. This, however, occurs only when the covariances of observations and odometries are set to be identity matrices, although the resulting solution is very close to the true solution obtained using the correct sensor and motion models. A number of numerical experiments demonstrated that the chance of getting trapped in a local minimum from a random initial guess is only about 20%. The DLR-Spatial Cognition dataset [6] also exhibits similar behavior, when started from a zero initial guess.

These results indicate that the number of local minima present in the nonlinear least squares formulation of the SLAM problem is likely to be small if the covariance matrices are spherical. This observation is the main motivation for the work presented in this paper. In particular, we examine the problem of joining two maps as well as the special case where information gathered at two robot poses are combined to build a local map. We argue that any feature based SLAM problem can be decomposed into a sequence containing these two steps. It is theoretically proven that the nonlinear least squares optimization problems associated with both these scenarios have at most two local minima. It is experimentally demonstrated that (a) the two local minima occur only when the odometry and observation information are extremely inconsistent with each other, and (b) the solution to the approximate map joining problem using spherical covariance matrices is practically very close to the true solution to the map joining problem using the actual covariance matrices.

The paper is organized as follows. Section II formulates the least squares SLAM and map joining problems. Section III provides a lemma which underpins the proofs presented. Section IV analyzes the one step SLAM problem while Section V examines the map joining problem. Experimental results to demonstrate the outcomes of the analysis is presented in Section VI. Section VII concludes the paper and Appendix presents some of the detailed proofs.

II. LEAST SQUARES SLAM AND MAP JOINING FORMULATION

The two-dimensional point feature based SLAM problem can be formulated in two ways as follows.

A. Least Squares SLAM Formulation

Suppose a number of 2D point features f_1, f_2, \dots, f_N in the environments are observed from a sequence of 2D

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robot poses $r_0, r_1, r_2, \dots, r_p$. The first robot pose (pose r_0) is chosen as the origin of the global coordinate frame.

We use $X_{f_j} = (x_{f_j}, y_{f_j})^T$ to denote the x, y position of feature f_j . $X_{r_i} = (x_{r_i}, y_{r_i})^T$ denotes the x, y position of robot pose r_i while ϕ_{r_i} denotes the orientation of pose r_i . R_{r_i} is the rotation matrix of pose r_i given by

$$R_{r_i} = R(\phi_{r_i}) = \begin{bmatrix} \cos \phi_{r_i} & -\sin \phi_{r_i} \\ \sin \phi_{r_i} & \cos \phi_{r_i} \end{bmatrix}. \quad (1)$$

The least squares SLAM formulation [7] is to use the odometry and observation information to estimate the state vector containing all the robot poses and all the feature positions¹

$$X = (X_{f_1}, \dots, X_{f_N}, X_{r_1}, \phi_{r_1}, \dots, X_{r_p}, \phi_{r_p}) \quad (2)$$

and the SLAM problem is to minimize [7]

$$F(X) = \sum_{i=1}^p (O_i^{i-1} - H^{O_i}(X))^T P_{O_i}^{-1} (O_i^{i-1} - H^{O_i}(X)) + \sum_{i,j} (Z_j^i - H^{Z_j^i}(X))^T P_{Z_j^i}^{-1} (Z_j^i - H^{Z_j^i}(X)) \quad (3)$$

where O_i^{i-1} ($1 \leq i \leq p$) are odometries, Z_j^i are observations, and P_{O_i} and $P_{Z_j^i}$ are the corresponding covariance matrices.

In the above least squares SLAM formulation, $H^{Z_j^i}(X)$ and $H^{O_i}(X)$ are the corresponding functions relating Z_j^i and O_i^{i-1} to the state X . The odometry is a function of two poses ($X_{r_{i-1}}, \phi_{r_{i-1}}$) and (X_{r_i}, ϕ_{r_i}) and is given by

$$H^{O_i}(X) = \begin{bmatrix} R_{r_{i-1}}^T (X_{r_i} - X_{r_{i-1}}) \\ \phi_{r_i} - \phi_{r_{i-1}} \end{bmatrix}. \quad (4)$$

The observation is a function of one pose (X_{r_i}, ϕ_{r_i}) and one feature position X_{f_j} and is given by

$$H^{Z_j^i}(X) = R_{r_i}^T (X_{f_j} - X_{r_i}). \quad (5)$$

In particular, since $\phi_{r_0} = 0$ and $X_{r_0} = (0, 0)^T$, the odometry function from robot r_0 to r_1 is given by

$$H^{O_1}(X) = \begin{bmatrix} X_{r_1} \\ \phi_{r_1} \end{bmatrix} \quad (6)$$

and the observation function from robot r_0 to f_j is given by

$$H^{Z_j^0}(X) = X_{f_j}. \quad (7)$$

“One step SLAM” problem is the special case where the number of robot poses is two, i.e. $p = 1$.

B. Map Joining

Joining of multiple local maps obtained by solving the above least squares problem can also be formulated as an optimization problem [8][13]. Suppose that there are a sequence of k local maps. The end robot pose of local map j is the start robot pose of local map $j+1$. The state vector of the map joining problem considered in [8] contains all the feature positions and robot end poses of each local map:

$$X_{MJ} = (X_{r_{1e}}, \phi_{r_{1e}}, \dots, X_{r_{ke}}, \phi_{r_{ke}}, X_{f_1}, \dots, X_{f_N}) \quad (8)$$

¹To simplify the notation, sometimes the transpose is omitted.

where r_{je} is the robot end pose of local map j ($1 \leq j \leq k$).

Suppose local map j is defined by (\hat{X}_j^L, P_j^L) where \hat{X}_j^L is the state estimate and P_j^L is the associated covariance matrix. Also assume the features present in the local map j are f_{j1}, \dots, f_{jn} . The local map state estimate \hat{X}_j^L can be regarded as an observation of the true relative positions from the robot start pose $X_{r_{(j-1)e}}, \phi_{r_{(j-1)e}}$ to the features $X_{f_{j1}}, \dots, X_{f_{jn}}$ and the robot end pose $X_{r_{je}}, \phi_{r_{je}}$. That is,

$$\hat{X}_j^L = H_j(X_{MJ}) + w_j \quad (9)$$

where

$$H_j(X_{MJ}) = \begin{pmatrix} R_{r_{(j-1)e}}^T (X_{r_{je}} - X_{r_{(j-1)e}}) \\ \phi_{r_{je}} - \phi_{r_{(j-1)e}} \\ R_{r_{(j-1)e}}^T (X_{f_{j1}} - X_{r_{(j-1)e}}) \\ \vdots \\ R_{r_{(j-1)e}}^T (X_{f_{jn}} - X_{r_{(j-1)e}}) \end{pmatrix}$$

and w_j is the zero-mean Gaussian “observation noise” whose covariance matrix is P_j^L (when $j = 1$, $X_{r_{(j-1)e}} = [0, 0]^T, \phi_{r_{(j-1)e}} = 0$).

So the problem of joining local maps 1 to k is to estimate the global state X_{MJ} using all the local map information (9) for $j = 1, \dots, k$. This problem can be formulated as a least squares problem. That is, finding X_{MJ} such that

$$\sum_{j=1}^k (\hat{X}_j^L - H_j(X_{MJ}))^T (P_j^L)^{-1} (\hat{X}_j^L - H_j(X_{MJ})) \quad (10)$$

is minimized.

Most of the map joining algorithms such as sequential map joining [8][9][10][11] and divide-and-conquer strategy [12][13] combine two local map at a time. Furthermore, it can be seen that “one step SLAM” problem defined in (3) with $p = 1$ is also a special case of joining two maps. Therefore, any feature based SLAM problem can be decomposed to a sequence of map joining problems.

III. A USEFUL LEMMA

It will be shown in the following sections that the problem of joining two maps and its special case, one step SLAM problem can both be reduced to a nonlinear equation constrained by a nonlinear inequality with one variable, when associated uncertainties can be described using spherical covariance matrices. The following lemma gives a special property of such problems.

Lemma 1: Assume that $a > 0$, $C_\phi \in [-\pi, \pi)$ are two scalars. Consider the following two conditions:

$$f(\phi) = a \sin(\phi + C_\phi) + \phi = 0 \quad (11)$$

$$g(\phi) = a \cos(\phi + C_\phi) + 1 > 0 \quad (12)$$

There are at least one and at most two $\phi \in [-\pi, \pi)$ satisfying (11)-(12) simultaneously. Moreover, there are two solutions

if and only if

$$a \geq 1 \quad (13)$$

$$-a \sin C_\phi - \pi \leq 0 \quad (14)$$

$$\sqrt{a^2 - 1} + \phi_1 \geq 0 \quad (15)$$

$$-\sqrt{a^2 - 1} + \phi_2 \leq 0 \quad (16)$$

$$-a \sin C_\phi + \pi \geq 0 \quad (17)$$

$$\phi_1 - \phi_2 \leq 0 \quad (18)$$

hold. Here

$$\begin{aligned} \phi_1 &= \text{wrap}(\arccos(-\frac{1}{a}) - C_\phi) \\ \phi_2 &= \text{wrap}(-\arccos(-\frac{1}{a}) - C_\phi) \end{aligned} \quad (19)$$

where $\text{wrap}(\theta)$ is a function which wraps θ into $[-\pi, \pi)$.

Proof: The proof is omitted due to space limitations. The MATLAB source code for testing the lemma is available at <http://services.eng.uts.edu.au/~sdhuang/research.htm>. ■

Remark 1: Fig. 1 illustrates the conditions (13)-(18). The possible pair of a, C_ϕ when there are two solutions to satisfy conditions (11)-(12) simultaneously is shown in the shaded area. For example, it can be seen that if $a < 1$, there is only one solution. If $|C_\phi| < \arcsin(\frac{\pi}{\sqrt{\pi^2+1}}) = 1.2626$, there is also only one solution. Fig. 2 shows the functions $f(\phi)$ and $g(\phi)$ when $a = 3, C_\phi = 2$ and it is clear that there are two solutions to (11)-(12).

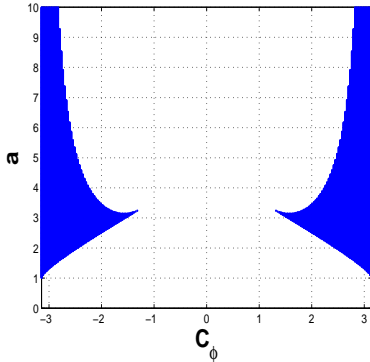


Fig. 1. Possible situations of having two solutions by satisfying conditions (13)-(18). The x-axis is C_ϕ , and y-axis is a . In the shaded area, there are two solutions to (11)-(12), in the other area, there is only one solution.

IV. ONE STEP SLAM

This section analyzes the number of local minima present in the one step SLAM problem.

Suppose there are n features which are all observed by both pose r_0 and pose r_1 , as shown in Fig. 3. Denote $X = (x_{f_1}, y_{f_1}, \dots, x_{f_n}, y_{f_n}, x_r, y_r, \phi)$, and consider the case when the covariance matrices $P_{O_1}, P_{Z_i^0}, P_{Z_i^1}$ are all identity matrices.

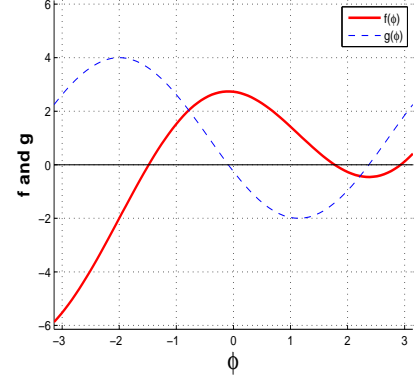


Fig. 2. An example of two solutions to (11)-(12): $a = 3, C_\phi = 2$.

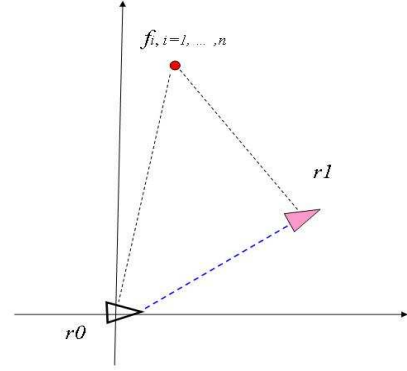


Fig. 3. One step SLAM problem with n feature

The SLAM problem is to minimize

$$\begin{aligned} f(X) &= (O_1^0 - H^{O_1}(X))^T P_{O_1}^{-1} (O_1^0 - H^{O_1}(X)) \\ &+ \sum_{i=1}^n (Z_i^0 - H^{Z_i^0}(X))^T P_{Z_i^0}^{-1} (Z_i^0 - H^{Z_i^0}(X)) \\ &+ \sum_{i=1}^n (Z_i^1 - H^{Z_i^1}(X))^T P_{Z_i^1}^{-1} (Z_i^1 - H^{Z_i^1}(X)) \\ &= (z_{x_r} - x_r)^2 + (z_{y_r} - y_r)^2 + (z_\phi - \phi)^2 \\ &+ \sum_{i=1}^n (z_{x_{f_i}} - x_{f_i})^2 + (z_{y_{f_i}} - y_{f_i})^2 \\ &+ \sum_{i=1}^n (Z_i^1 - R(\phi)^T \delta_i)^T (Z_i^1 - R(\phi)^T \delta_i) \end{aligned} \quad (20)$$

where $O_1^0 = (z_{x_r}, z_{y_r}, z_\phi)^T$ is the odometry between r_0 and r_1 , and $Z_i^0 = (z_{x_{f_i}}, z_{y_{f_i}})^T$ is the observation of f_i from r_0 , $Z_i^1 = (z_{x_{f_i}}, z_{y_{f_i}})^T$ is the observation of f_i from r_1 , and

$$\delta_i = X_{f_i} - X_r = \begin{bmatrix} x_{f_i} - x_r \\ y_{f_i} - y_r \end{bmatrix} \quad (21)$$

Note that

$$\begin{aligned} &(Z_i^1 - R(\phi)^T \delta_i)^T (Z_i^1 - R(\phi)^T \delta_i) \\ &= |Z_i^1 - R(\phi)^T \delta_i|^2 \\ &= |R(\phi) Z_i^1 - \delta_i|^2 \end{aligned} \quad (22)$$

Thus the objective function (20) can be converted into

$$\begin{aligned}
f(X) &= (z_{x_r} - x_r)^2 + (z_{y_r} - y_r)^2 + (z_\phi - \phi)^2 \\
&+ \sum_{i=1}^n [(z_{x_{f_i}} - x_{f_i})^2 + (z_{y_{f_i}} - y_{f_i})^2] \\
&+ \sum_{i=1}^n [(A_i - (x_{f_i} - x_r))^2 + (B_i - (y_{f_i} - y_r))^2]
\end{aligned} \quad (23)$$

where

$$A_i = z_{x_{f_i}}^r c_\phi - z_{y_{f_i}}^r s_\phi, \quad B_i = z_{x_{f_i}}^r s_\phi + z_{y_{f_i}}^r c_\phi. \quad (24)$$

Here c_ϕ, s_ϕ denote $\cos \phi$ and $\sin \phi$, respectively. Note that A_i and B_i satisfy the following equations

$$\frac{dA_i}{d\phi} = -B_i, \quad \frac{dB_i}{d\phi} = A_i, \quad A_i^2 + B_i^2 = z_{x_{f_i}}^2 + z_{y_{f_i}}^2. \quad (25)$$

The number of local minima of objective function (23) is given by Theorem 1.

Theorem 1: The one step SLAM problem with n features has at least one local minimum and at most two local minima. Moreover, there are two local minima if and only if conditions (13)-(18) hold with

$$a = \sqrt{p^2 + (d+q)^2}, \quad C_\phi = \text{atan2}(p, d+q) \quad (26)$$

where $\text{atan2}(y, x)$ denotes the arc tangent of y, x and

$$\begin{aligned}
d &= \frac{1}{2} \sum_{1 \leq i \leq n} [(z_{x_{f_i}} - z_{x_r})^2 + (z_{y_{f_i}} - z_{y_r})^2] \\
&- \frac{1}{2(n+2)} \sum_{1 \leq i, j \leq n} [(z_{x_{f_j}} - z_{x_r})(z_{x_{f_i}} - z_{x_r}) \\
&+ (z_{y_{f_j}} - z_{y_r})(z_{y_{f_i}} - z_{y_r})]
\end{aligned} \quad (27)$$

$$\begin{aligned}
p &= \delta_a c_{z_\phi} + \delta_b s_{z_\phi} \\
q &= -\delta_a s_{z_\phi} + \delta_b c_{z_\phi}
\end{aligned} \quad (28)$$

with

$$\begin{aligned}
\delta_a &= \sum_{j=1}^n \left(\begin{bmatrix} \Delta z_{x_{f_j}}^r \\ \Delta z_{y_{f_j}}^r \end{bmatrix} \right)^T \begin{bmatrix} \frac{1}{n+2} z_{y_r} - \frac{1}{2} z_{y_{f_j}} + \frac{1}{2(n+2)} \sum_{i=1}^n z_{y_{f_i}} \\ \frac{-1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^n z_{x_{f_i}} \end{bmatrix} \\
\delta_b &= \sum_{j=1}^n \left(\begin{bmatrix} \Delta z_{x_{f_j}}^r \\ \Delta z_{y_{f_j}}^r \end{bmatrix} \right)^T \begin{bmatrix} \frac{-1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^n z_{x_{f_i}} \\ \frac{-1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^n z_{y_{f_i}} \end{bmatrix}
\end{aligned} \quad (29)$$

Here for $i = 1, \dots, n$,

$$\begin{aligned}
\Delta z_{x_{f_i}}^r &= z_{x_{f_i}}^r - [(z_{x_{f_i}} - z_{x_r})c_{z_\phi} + (z_{y_{f_i}} - z_{y_r})s_{z_\phi}] \\
\Delta z_{y_{f_i}}^r &= z_{y_{f_i}}^r - [-(z_{x_{f_i}} - z_{x_r})s_{z_\phi} + (z_{y_{f_i}} - z_{y_r})c_{z_\phi}]
\end{aligned}$$

where z_{x_r}, z_{y_r}, z_ϕ is the odometry, $z_{x_{f_i}}, z_{y_{f_i}}, z_{x_{f_i}}^r, z_{y_{f_i}}^r, i = 1, \dots, n$ are the observations, c_{z_ϕ}, s_{z_ϕ} denote $\cos(z_\phi)$ and $\sin(z_\phi)$, respectively.

Proof: See Appendix. \blacksquare

Remark 2: In the ideal case when all the sensors are perfect, we have $\Delta z_{x_{f_i}}^r = 0, \Delta z_{y_{f_i}}^r = 0, i = 1, \dots, n$. Then we have $p = 0, q = 0, C_\phi = 0$, and $a = d > 0$, it is evident that (18) does not hold anymore since $\frac{\pi}{2} < \arccos(-\frac{1}{a}) < \pi$, thus only one minimum exists which is the global optimal solution to the SLAM problem.

Numerical Illustration. Consider the special case where only one feature f is observed from the two poses. Assume the odometry is given by $(z_{x_r}, z_{y_r}, z_\phi)$ and the observation from pose r_0 to f is (z_{x_f}, z_{y_f}) . Let the numerical values of these observations be $(2, 2, 0.5, 0, 3)$. Then

$$\begin{aligned}
(z_{x_f} - z_{x_r})c_{z_\phi} + (z_{y_f} - z_{y_r})s_{z_\phi} &= -1.2757 \\
-(z_{x_f} - z_{x_r})s_{z_\phi} + (z_{y_f} - z_{y_r})c_{z_\phi} &= 1.8364
\end{aligned}$$

Thus, $z_x = -1.2757 + \Delta z_x$, and $z_y = 1.8364 + \Delta z_y$.

Given Δz_x and Δz_y , numerical values for $\delta_a, \delta_b, d, p, q, a, C_\phi$ can all be computed, and conditions (13)-(18) can be evaluated. Fig. 4 shows number of minima that exist for different $\Delta z_x, \Delta z_y$. The shaded area corresponds to the case where there are two local minima, while the remaining space corresponds to the conditions where there is only one minimum.

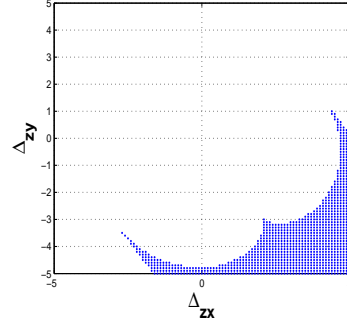


Fig. 4. Number of local minima to the one step one feature SLAM problem as a function of $\Delta z_x, \Delta z_y$. When $|\Delta z_x| \leq 3, |\Delta z_y| \leq 3$, there is only one local minimum. Normally, one cannot expect 3m measurement error with measurement values within 2m. So it is very unlikely to have two local minima.

V. JOINING TWO LOCAL MAPS

This section demonstrates that the least squares optimization problem of joining two local maps, also has at most two local minima.

Consider the two local maps shown in Fig. 5. n_1 is the number of features that appear only in map 1, n is the number of features that appear in both map 1 and map 2, and n_2 is the number of features that appear only in map 2. r_0 is the start pose of local map 1 which is the origin of the global map. r_1 is the end pose in local map 1 as well as the start pose of local map 2, r_2 is the end pose in local map 2.

When the uncertainties associated with the local maps are described using spherical covariance matrices, similar to (23), the map joining problem becomes minimizing the

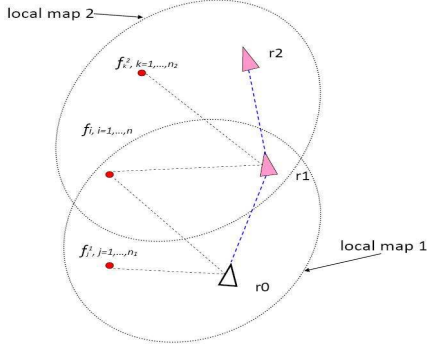


Fig. 5. Joining of two local maps

following objective function

$$\begin{aligned}
f(X_{MJ}) = & (z_{x_{r_1}} - x_{r_1})^2 + (z_{y_{r_1}} - y_{r_1})^2 + (z_{\phi_{r_1}} - \phi_{r_1})^2 \\
& + (A_r - (x_{r_2} - x_{r_1}))^2 + (B_r - (y_{r_2} - y_{r_1}))^2 + (z_{\phi_{r_2}} \\
& - (\phi_{r_2} - \phi_{r_1}))^2 + \sum_{j=1}^{n_1} [(z_{x_{f_j^1}} - x_{f_j^1})^2 + (z_{y_{f_j^1}} - y_{f_j^1})^2] \\
& + \sum_{i=1}^n [(z_{x_{f_i}} - x_{f_i})^2 + (z_{y_{f_i}} - y_{f_i})^2] \\
& + \sum_{i=1}^n [(A_i - (x_{f_i} - x_{r_1}))^2 + (B_i - (y_{f_i} - y_{r_1}))^2] \\
& + \sum_{k=1}^{n_2} [(C_k - (x_{f_k^2} - x_{r_1}))^2 + (D_k - (y_{f_k^2} - y_{r_1}))^2] \quad (30)
\end{aligned}$$

where state X_{MJ} contains $(x_{r_1}, y_{r_1}, \phi_{r_1})$, $(x_{r_2}, y_{r_2}, \phi_{r_2})$, and all the feature positions, and

$$\begin{aligned}
A_r &= c_{\phi_{r_1}} z_{x_{r_2}^{r_1}} - s_{\phi_{r_1}} z_{y_{r_2}^{r_1}}, B_r = s_{\phi_{r_1}} z_{x_{r_2}^{r_1}} + c_{\phi_{r_1}} z_{y_{r_2}^{r_1}} \\
A_i &= c_{\phi_{r_1}} z_{x_{f_i}^{r_1}} - s_{\phi_{r_1}} z_{y_{f_i}^{r_1}}, B_i = s_{\phi_{r_1}} z_{x_{f_i}^{r_1}} + c_{\phi_{r_1}} z_{y_{f_i}^{r_1}} \\
C_k &= c_{\phi_{r_1}} z_{x_{f_k^2}^{r_1}} - s_{\phi_{r_1}} z_{y_{f_k^2}^{r_1}}, D_k = s_{\phi_{r_1}} z_{x_{f_k^2}^{r_1}} + c_{\phi_{r_1}} z_{y_{f_k^2}^{r_1}}
\end{aligned}$$

Here $(z_{x_{r_1}}, z_{y_{r_1}}, z_{\phi_{r_1}})^T$ is the estimate of pose r_1 in local map 1. $(z_{x_{r_2}^{r_1}}, z_{y_{r_2}^{r_1}}, z_{\phi_{r_2}^{r_1}})^T$ is the estimate of pose r_2 in local map 2. $z_{x_{f_j^1}}, z_{y_{f_j^1}}, j = 1, \dots, n_1, z_{x_{f_i}}, z_{y_{f_i}}, z_{x_{f_i}^{r_1}}, z_{y_{f_i}^{r_1}}, i = 1, \dots, n, z_{x_{f_k^2}^{r_1}}, z_{y_{f_k^2}^{r_1}}, k = 1, \dots, n_2$ are the estimated positions of features in local map 1 and local map 2.

Theorem 2: The map joining problem with two local maps has at least one local minimum and at most two local minima. Moreover, there are two local minima if and only if conditions (13)-(18) hold with a, C_ϕ defined in (26), $d, p, q, \delta_a, \delta_b$ defined similar to (27), (28), and (29) with

$$\begin{aligned}
\Delta z_{x_{f_i}^{r_1}} &= z_{x_{f_i}^{r_1}} - [(z_{x_{f_i}} - z_{x_{r_1}})c_{z_{\phi_{r_1}}} + (z_{y_{f_i}} - z_{y_{r_1}})s_{z_{\phi_{r_1}}}] \\
\Delta z_{y_{f_i}^{r_1}} &= z_{y_{f_i}^{r_1}} - [-(z_{x_{f_i}} - z_{x_{r_1}})s_{z_{\phi_{r_1}}} + (z_{y_{f_i}} - z_{y_{r_1}})c_{z_{\phi_{r_1}}}] \\
i &= 1, \dots, n
\end{aligned}$$

Proof: The proof follows similar arguments to those used for proving Theorem 1. It should be noted that some data including the numbers n_1, n_2 do not affect the results. Detailed proof is omitted due to the space limitation. ■

Remark 3: Both sequential map joining [8][9] and divide-and-conquer map joining [12] combine two local maps at a time. Thus Theorem 2 presents an important result related to existing map joining algorithms. The major simplification is that the local map covariance matrix is assumed to be identity.

Remark 4: Theorems 1 and 2 can be extended to the case where covariance matrices for each observation/odometry (feature/pose) are all spherical but different from each other.

Remark 5: It is easy to see that when combining two local maps each containing more than two robot poses (e.g. Tectonic SAM map joining [13]), the same results hold as long as the covariances are spherical.

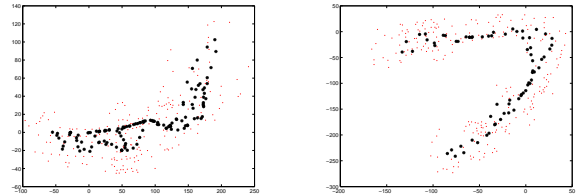
Remark 6: If the robot poses are not available, map joining problem reduces to the problem of finding the relative transformation between two coordinate frames given two corresponding point sets. When the covariances of the uncertainty associated with feature locations are assumed to be spherical, it is known that the problem has a closed form solution [14].

VI. EXPERIMENTAL RESULTS

In this section, we use publicly available experimental datasets to demonstrate that the problem of joining two local maps has only one local minima and that the map joining solution obtained under the assumption of spherical covariance matrices is close to the true optimal solution using the original covariances².

A. Results using Victoria Park dataset

The Victoria Park dataset was divided into two parts to build two local maps which are shown in Fig. 6. The covariance matrices of the two local maps were set to identity matrices. Using the local map data to compute the values of a, C_ϕ in Theorem 2, we obtain $a = 203660, C_\phi = -0.0029$. Obviously they do not satisfy the conditions (13)-(18), meaning that the map joining problem only has one local minimum. To check the result, Gauss Newton algorithm is used to solve the map joining problem. In an experiment with more than 100 trials, the algorithm always converged to the solution shown in Fig. 7(b) from arbitrary initial guesses to the robot poses and feature locations. Example initial guess is shown in Fig. 7(a).

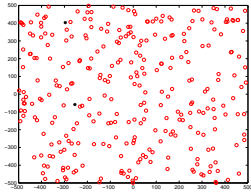


(a) Local map 1.

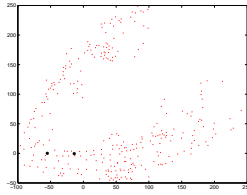
(b) Local map 2.

Fig. 6. Local maps 1 and 2 of Victoria Park dataset. The black stars denote the robot position, the red dots denote the features.

²The MATLAB source code used in this section is available at <http://services.eng.uts.edu.au/~sdhuan/research.htm>.



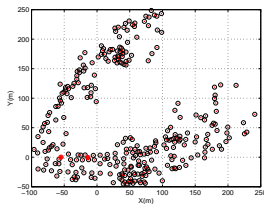
(a) Example initial guess to robot and feature locations.



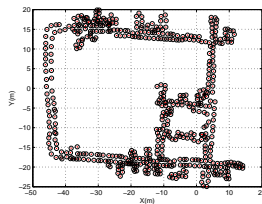
(b) Result of map joining.

Fig. 7. Result of map joining for the Victoria Park dataset. The black stars denote the robot position, the red circles denote the features.

Fig. 8(a) compares the map joining result using spherical covariance matrices with that using the original covariance matrices of the two local maps, the differences due to the use of spherical covariance matrices is negligible.



(a) Victoria Park dataset

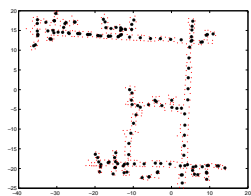


(b) DLR dataset

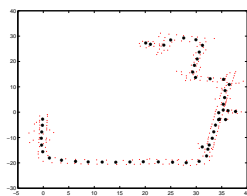
Fig. 8. Comparison of global map using spherical covariance matrix and original covariances by combining the two local maps. The two results are almost identical.

B. Results using DLR-Spatial Cognition dataset

The experiments described in Section VI-A was repeated using the DLR dataset. Fig. 9 shows the two local maps obtained. An example initial guess and the map joining results are shown in Fig. 10(a) and Fig. 10(b). Fig. 8(b) compares the map joining result using identity covariance matrices with that using the original covariance matrices.



(a) Local map 1.

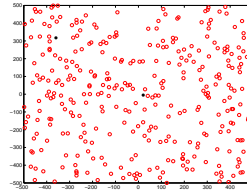


(b) Local map 2.

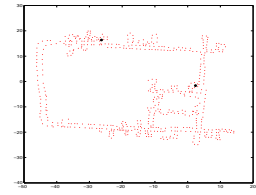
Fig. 9. Local maps 1 and 2 of DLR dataset. The black stars denote the robot position, the red dots denote the features.

VII. CONCLUSIONS AND FUTURE WORK

The main contribution of this paper is a mathematical proof of special properties of the map joining problem in SLAM. Nonlinear least squares problem associated with joining two maps with spherical covariance matrices is shown to have at most two local minima. Moreover, it is demonstrated that two local minima exist only if the quality



(a) Arbitrary initial state.



(b) Result of map joining.

Fig. 10. Arbitrary initial state and result of map joining of the DLR dataset. The black stars denote the robot position, the red circles denote the features.

of local maps are much poorer than what is practically achievable. This paper also provides the conditions for the existence of two local minima, which can be evaluated using the two local map data. This makes it possible to guarantee that the globally optimum solution has been reached leading to the possibility of obtaining robust solutions to the SLAM problem even when the initial guess obtained through odometry or relative pose estimates is unreliable.

Impact of the results presented is threefold. First, it is clear that the joining of two maps with spherical covariance matrices can be simplified to that of solving a constrained nonlinear equation of just one variable. Given the argument that all SLAM problems can be decomposed to that of joining two maps, it may be possible to use simple techniques such as bisection to obtain a solution to SLAM very efficiently. However, further work is needed to evaluate the impact of the assumption of spherical covariance matrices. Second, results presented in this paper clearly show that SLAM is a very special optimization problem, and goes somewhat towards explaining the success of some of the recent techniques for SLAM that rely on local search strategies yet lead to good solutions. Further work on the analysis of these algorithms, for example TORO, may lead to even better and more efficient solutions to SLAM. Third, an experimental demonstration confirms that the covariance structure has only a small influence on the final solution to the SLAM problem. For the Victoria Park and DLR datasets, the solutions obtained using spherical covariances are close to those using the covariances computed from the sensor model. This is somewhat unexpected and a key question is to ask why and also determine the conditions under which this occurs. Work in all three directions has the potential to enhance the understanding of this important robotics problem and lead to more reliable and efficient solutions to robot navigation.

APPENDIX

This appendix provides the proof of Theorem 1..... (temporarily removed)..... please contact the first author if you want the details.

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