

Understanding Extended Kalman Filter – Part III: Extended Kalman Filter

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Abstract

Kalman Filter (KF) and extended Kalman Filter (EKF) are basic tools for solving many estimation problems. They have found plenty of applications including target tracking and mobile robot localization and mapping. In this note, I tried to explain the KF and EKF formulas without using too much knowledge of probability theory. In Part I, one dimensional KF formula is derived. In Part II, multi dimensional KF formula is proven (you should have read these two parts already). In Part III (this note), EKF formula is proven.

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1 Introduction

I assume the readers have already read the Part I and Part II of this notes (one dimensional KF and multi dimensional KF).

The structure of this note is the following. The extended Kalman Filter (EKF) formula is reviewed in Section 2. The derivation of EKF prediction formulas are explained in Section 3 and the EKF update formulas are derived in Section 4. A question is given in Section 5 to check whether you have completely understand EKF formulas or not.

2 Extended Kalman Filter formula

Different from KF, EKF deals with nonlinear process model and nonlinear observation model. The nonlinear process model (from time k to time $k + 1$) is described as

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k, \quad (1)$$

where $\mathbf{x}_k, \mathbf{x}_{k+1}$ are the system state (vector) at time $k, k + 1$, \mathbf{f} is the system transition function, \mathbf{u}_k is the control, and \mathbf{w}_k is the zero-mean Gaussian process noise $\mathbf{w}_k \sim N(0, Q)$.

For state estimation problem, the true system state is not available and needs to be estimated. The initial state x_0 is assumed to follow a known Gaussian distribution $\mathbf{x}_0 \sim N(\hat{\mathbf{x}}_0, P_0)$. The objective is to estimate the state at each time step by the process model and the observations.

The observation model at time $k + 1$ is given by

$$\mathbf{z}_{k+1} = \mathbf{h}(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1}. \quad (2)$$

where \mathbf{h} is the observation function and \mathbf{v}_{k+1} is the zero-mean Gaussian observation noise $\mathbf{v}_{k+1} \sim N(0, R)$.

Suppose the knowledge on \mathbf{x}_k at time k is

$$\mathbf{x}_k \sim N(\hat{\mathbf{x}}_k, P_k), \quad (3)$$

then \mathbf{x}_{k+1} at time $k + 1$ follows

$$\mathbf{x}_{k+1} \sim N(\hat{\mathbf{x}}_{k+1}, P_{k+1}) \quad (4)$$

where $\hat{\mathbf{x}}_{k+1}, P_{k+1}$ can be computed by the following **Extended Kalman Filter** formula.

Predict using process model:

$$\bar{\mathbf{x}}_{k+1} = \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k) \quad (5)$$

$$\bar{P}_{k+1} = \nabla \mathbf{f}_{\mathbf{x}} P_k \nabla \mathbf{f}_{\mathbf{x}}^T + Q \quad (6)$$

where $\nabla \mathbf{f}_{\mathbf{x}}$ is the Jacobian of function \mathbf{f} with respect to \mathbf{x} evaluated at $\hat{\mathbf{x}}_k$.

Update using observation:

$$\hat{\mathbf{x}}_{k+1} = \bar{\mathbf{x}}_{k+1} + K(\mathbf{z}_{k+1} - \mathbf{h}(\bar{\mathbf{x}}_{k+1})) \quad (7)$$

$$P_{k+1} = \bar{P}_{k+1} - KSK^T, \quad (8)$$

where the innovation covariance S (here $\mathbf{z}_{k+1} - \mathbf{h}(\bar{\mathbf{x}}_{k+1})$ is called innovation) and the Kalman gain K are given by

$$S = \nabla \mathbf{h} \bar{P}_{k+1} \nabla \mathbf{h}^T + R \quad (9)$$

$$K = \bar{P}_{k+1} \nabla \mathbf{h}^T S^{-1}. \quad (10)$$

where $\nabla \mathbf{h}$ is the Jacobian of function \mathbf{h} evaluated at $\bar{\mathbf{x}}_{k+1}$.

Remark 2.1 If you just want to apply extended Kalman Filter, then the above formulas are enough as long as you have got your process model and observation model. If you would like to know where these formulas come from, then proceed to the following sections.

3 EKF prediction

This section shows that the EKF prediction formula can be obtained easily by linearizing the process model and applying the KF prediction formula.

3.1 Linearization of process model

The major difference between KF and EKF is that KF deals with linear model and EKF deals with nonlinear model. Thus linearization is a key step for deriving EKF formulas.

The process model (1) is linearized at the current estimate $\hat{\mathbf{x}}_k$ using the first order Taylor series expansion,

$$\mathbf{x}_{k+1} \approx \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k) + \nabla \mathbf{f}_{\mathbf{x}}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{w}_k \quad (11)$$

where $\nabla \mathbf{f}_{\mathbf{x}}$ is the Jacobian of function \mathbf{f} with respect to \mathbf{x} evaluated at $\hat{\mathbf{x}}_k$. In this equation, all the higher order terms are ignored since we assume \mathbf{x}_k is close to $\hat{\mathbf{x}}_k$.

System (11) is a linear process model

$$\mathbf{x}_{k+1} \approx F\mathbf{x}_k + U_k + \mathbf{w}_k \quad (12)$$

with $F = \nabla \mathbf{f}_{\mathbf{x}}$ and $U_k = \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k) - \nabla \mathbf{f}_{\mathbf{x}}\hat{\mathbf{x}}_k$.

3.2 Deriving the EKF prediction formula

Now applying the linear KF prediction formula using the linear process model (12), we have

$$\bar{\mathbf{x}}_{k+1} = F\hat{\mathbf{x}}_k + U_k = \nabla \mathbf{f}_{\mathbf{x}}\hat{\mathbf{x}}_k + \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k) - \nabla \mathbf{f}_{\mathbf{x}}\hat{\mathbf{x}}_k = \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k) \quad (13)$$

$$\bar{P}_{k+1} = FP_kF^T + Q = \nabla \mathbf{f}_{\mathbf{x}}P_k\nabla \mathbf{f}_{\mathbf{x}}^T + Q. \quad (14)$$

These are (5) and (6).

4 EKF update

This sections shows that the EKF update formula can be obtained easily by linearizing the observation model and applying the KF update formula.

4.1 Linearization of observation model

After the prediction step, the current estimate of the state is $\bar{\mathbf{x}}_{k+1}$.

The observation model (2) is linearized at the current estimate $\bar{\mathbf{x}}_{k+1}$ using first order Taylor series expansion,

$$\mathbf{z}_{k+1} \approx \mathbf{h}(\bar{\mathbf{x}}_{k+1}) + \nabla \mathbf{h}(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}) + \mathbf{v}_{k+1}, \quad (15)$$

where $\nabla \mathbf{h}$ is the Jacobian of function \mathbf{h} evaluated at $\bar{\mathbf{x}}_{k+1}$. Here the higher order terms are ignored since we assume that \mathbf{x}_{k+1} is close to $\bar{\mathbf{x}}_{k+1}$.

The observation model (15) is a linear observation model

$$\mathbf{Z}_{k+1} \approx H\mathbf{x}_{k+1} + \mathbf{v}_{k+1}, \quad (16)$$

where $H = \nabla \mathbf{h}$ and $\mathbf{Z}_{k+1} = \mathbf{z}_{k+1} - \mathbf{h}(\bar{\mathbf{x}}_{k+1}) + \nabla \mathbf{h}\bar{\mathbf{x}}_{k+1}$.

4.2 Deriving the EKF update formula

Now applying the linear KF update formula using the linear observation model (16), we have

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \bar{\mathbf{x}}_{k+1} + K(\mathbf{Z}_{k+1} - H\bar{\mathbf{x}}_{k+1}) \\ &= \bar{\mathbf{x}}_{k+1} + K(\mathbf{z}_{k+1} - \mathbf{h}(\bar{\mathbf{x}}_{k+1}) + \nabla \mathbf{h}\bar{\mathbf{x}}_{k+1} - \nabla \mathbf{h}\bar{\mathbf{x}}_{k+1}) \\ &= \bar{\mathbf{x}}_{k+1} + K(\mathbf{z}_{k+1} - \mathbf{h}(\bar{\mathbf{x}}_{k+1})) \end{aligned} \quad (17)$$

and

$$P_{k+1} = \bar{P}_{k+1} - KSK^T, \quad (18)$$

where the innovation covariance S and the Kalman gain K are given by

$$S = H\bar{P}_{k+1}H^T + R = \nabla \mathbf{h}\bar{P}_{k+1}\nabla \mathbf{h}^T + R \quad (19)$$

$$K = \bar{P}_{k+1}H^T S^{-1} = \bar{P}_{k+1}\nabla \mathbf{h}^T S^{-1}. \quad (20)$$

These are (7) and (8).

5 A question

The following question can be used to check whether you have completely understand the EKF formula or not.

Question: If the process model is given by

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k) \quad (21)$$

and the observation model is given by

$$\mathbf{z}_{k+1} = \mathbf{h}(\mathbf{x}_{k+1}, \mathbf{v}_{k+1}), \quad (22)$$

then what the EKF prediction and update formulas are? Can you derive them using linearization and the KF formulas?

Remark 5.1 If you can derive the EKF formula for the above general process model and observation model, then congratulations to you. You now understand KF and EKF as good as I do. In the future, you can teach me some other techniques.