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Analysis of Input to State Stability for Discrete Time Nonlinear Systems via Dynamic Programming*

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***Abstract*— This paper presents novel analysis results for input-to-state stability (ISS) that utilise dynamic programming techniques to characterise minimal ISS gains and transient bounds. These characterisations naturally lead to computable necessary and sufficient conditions for ISS. Our results make a connection between ISS and optimisation problems in nonlinear dissipative systems theory (including L_2-gain analysis and nonlinear H_∞ theory).**

I. INTRODUCTION

Among the many stability properties for systems with disturbances that have been proposed in the literature, the input-to-state stability (ISS) property proposed by Sontag in 1989 [13] deserves special attention. Indeed, ISS is fully compatible with Lyapunov stability theory [15] while its other equivalent characterizations relate it to robust stability, dissipativity and input-output stability theory [14], [16], [19]. The ISS property has found its main application in the ISS small gain theorem that was first proved by Jiang, Teel and Praly in [10]. Several different versions of the ISS small gain theorem that use different (equivalent) characterizations of the ISS property and their various applications to nonlinear controller design can be found in [11], [12], [20] and references defined therein.

The ISS property and the ISS small gain theorems naturally lead to the concept of nonlinear disturbance gain functions or simply "nonlinear gains". In this context, obtaining sharp estimates for the nonlinear gains is an important issue. Indeed, the better the nonlinear gain estimate that we can obtain, the larger the class of systems to which the ISS small gain results can be applied. Currently, the main tool for estimating the nonlinear gains are the so called ISS Lyapunov functions that typically produce rather conservative estimates (over bounds) for the ISS nonlinear gains.

It is the main purpose of this paper to present several results that provide a computational framework based on dynamic programming for obtaining *minimum ISS nonlinear gains*. These results are related to optimization based methods in nonlinear dissipative systems theory, such as L_2-gain analysis and nonlinear H_∞ theory (see [5] and references defined therein), as well as recently developed optimization based L_∞ methods (see [3], [6]).

The paper is organised as follows. In Section II we present several equivalent definitions of the ISS property and state a result from the literature that motivates our definitions and results. A fundamental dynamic programming equation that we need to state our main results is given in Section III. Section IV contains results on minimum nonlinear gains for different equivalent definitions of the ISS property. Several illustrative examples are presented in Section V and the paper is closed with conclusions in Section VI. All the proofs are omitted due to the space limitation. For a full version of this paper please refer to [7].

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II. PRELIMINARIES

Sets of real numbers, integers and nonnegative integers are denoted respectively as $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}_+$. A function $\gamma : [0,\infty) \to [0,\infty)$ is of class $\bar{\mathcal{K}}$ if it is nondecreasing, satisfies $\gamma(0) = 0$ and is right continuous at 0. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\bar{\mathcal{K}}$ and for each fixed $s \geq 0$, $\lim_{t\to+\infty} \beta(s,t) = 0$. Denote $l_\infty = \{u : \mathbf{Z}_+ \to \mathbf{R}^m : \|u\|_\infty = \sup_{k\in\mathbf{Z}_+} |u_k| < \infty\}$ where $|\cdot|$ is the Euclidean norm.

Consider the following dynamical system

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathbf{R}^n, u_k \in \mathbf{R}^m$, and $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is continuous and satisfies $f(0,0) = 0$. For any $x_0 \in \mathbf{R}^n$ and any input u, we denote by $x(\cdot, x_0, u)$ the solution of (1) with initial state x_0 and input u.

The following definitions are taken from ISS related literature. It was shown in [8] that these definitions of ISS are qualitatively equivalent. However, the gains in different definitions are not the same and since we are interested in minimum disturbance gains for different characterizations, we find it useful to introduce different notation for each of the different characterizations.

In all the definitions below we assume that $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

Definition 2.1: (Input-to-state stability with $+$ formulation) System (1) is ISS$_+$ (with (β, γ)) if

$$|x(k, x_0, u)| \leq \beta(|x_0|, k) + \gamma(\|u\|_\infty) \qquad (2)$$

for all $x_0 \in \mathbf{R}^n$, all $u \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Definition 2.2: (Asymptotic gain property) System (1) is AG (with gain γ) if for all $x_0 \in \mathbf{R}^n$ and all $u \in l_\infty$,

$$\limsup_{k \to +\infty} |x(k, x_0, u)| \leq \gamma(\|u\|_\infty). \qquad (3)$$

Definition 2.3: (Zero global asymptotic stability property) System (1) is 0-GAS (with β) if the state trajectories with $u \equiv 0$ satisfy

$$|x(k, x_0, 0)| \leq \beta(|x_0|, k). \qquad (4)$$

for all $x_0 \in \mathbf{R}^n$ and all $k \in \mathbf{Z}_+$.

Definition 2.4: (Input-to-state stability with asymptotic gain formulation) The system (1) is ISS$_{AG}$ (with (β, γ)) if it is AG (with gain γ) and 0-GAS (with β).

Remark 2.5: The above definition is motivated by the result proved in [16] which shows for continuous-time systems that $ISS_+ \Leftrightarrow AG + 0\text{-}GAS$. A similar result for discrete-time systems was proved in [4], [8]. This result is restated below in Theorem 2.8 for convenience.

Definition 2.6: (Input-to-state stability with $\max$ formulation) System (1) is ISS$_{\max}$ (with (β, γ)) if

$$|x(k, x_0, u)| \leq \max\{\beta(|x_0|, k), \gamma(\|u\|_\infty)\} \qquad (5)$$

for all $x_0 \in \mathbf{R}^n$, all $u \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Remark 2.7: It is more common in the literature to use the classes $\mathcal{K}$ and $\mathcal{KL}$ of functions when defining ISS and related properties. A function $\gamma : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$ $\beta(s, \cdot)$ decreases to zero.

It is not hard to see that the stability definitions that we use are qualitatively equivalent to the stability definitions when the classes of functions $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$ are replaced respectively by $\mathcal{K}$ and $\mathcal{KL}$. This follows from the following three facts: (i) $\mathcal{K} \subset \bar{\mathcal{K}}$ and $\mathcal{KL} \subset \bar{\mathcal{K}}\bar{\mathcal{L}}$; (ii) given any $\gamma \in \bar{\mathcal{K}}$, there exists $\gamma_1 \in \mathcal{K}$ such that $\gamma(s) \leq \gamma_1(s), \forall s \geq 0$; (iii) given any $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, there exists $\beta_1 \in \mathcal{KL}$ such that $\beta(s, k) \leq \beta_1(s, k), \forall s \geq 0, \forall k \in \mathbf{Z}_+$. Consequently, most results that were proved in the literature for classes of functions $\mathcal{K}$ and $\mathcal{KL}$ are still true when stated with function classes $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$.

The following theorem has been proved in the context of function classes $\mathcal{K}$ and $\mathcal{KL}$ for continuous-time systems in [16] and for discrete-time systems in [4], [8]. However, this result remains valid for function classes $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$.

Theorem 2.8: The following statements are equivalent:

1) There exist $\beta_{AG} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{AG} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{AG}$ with $(\beta_{AG}, \gamma_{AG})$;
2) There exist $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_+ \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_+$ with (β_+, γ_+);
3) There exist $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{\max} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$.

In the sequel we use the non-standard notation from Theorem 2.8 since it is important to distinguish between different characterizations and the related functions. Indeed, the functions $\beta_{AG}, \beta_+, \beta_{\max}$ (respectively functions $\gamma_{AG}, \gamma_+, \gamma_{\max}$) in the above theorem are all different in general. Note that although notation β_{AG} characterizing 0-GAS seems counterintuitive, it is consistent with the definition of ISS$_{AG}$ in Definition 2.4.

Remark 2.9: We note that each of the properties ISS$_{AG}$, ISS$_+$ and ISS$_{\max}$ has been used in the literature. In particular, there exist small gain theorems that use each of these different characterizations (see, for instance, [9], [10], [11], [12], [20]). Computing the smallest possible functions β, γ (or their estimates) in each of these properties is an important problem for the following reasons: (i) the smaller the estimates of gains functions, the larger the class of systems to which the small gain theorem can be applied; (ii) better estimates of the functions β, γ for subsystems produce (via the small gain theorems) sharper bounds on solutions of the composite system; (iii) the smallest functions will be different in general for each of the properties ISS$_{AG}$, ISS$_+$ and ISS$_{\max}$ (this further motivates our notation). In the sequel, we provide a framework for the computation of minimum functions $\beta_{AG}, \beta_+, \beta_{\max}$ and $\gamma_{AG}, \gamma_+, \gamma_{\max}$ via dynamic programming.

III. DYNAMIC PROGRAMMING

In this section we define a value function that is used in the derivation of our subsequent results, and present a dynamic programming equation to compute it.

For $x \in \mathbf{R}^n$, $\delta \geq 0$, integer $k \in \mathbf{Z}_+$, denote

$$V^\delta(x, k) := \sup_{\|u\|_\infty \leq \delta} \{|x(k, x_0, u)| : x_0 = x\}\ . \qquad (6)$$

The Dynamic Programming Equation (DPE) for $V^\delta(x, k)$ is

$$V^\delta(x, k) = \sup_{|u| \leq \delta} V^\delta(f(x, u), k - 1) \qquad (7)$$

with the initial condition

$$V^{\delta}(x,0)=|x|\,. \tag{8}$$

In the next section, we show how $V^{\delta}(x,k)$ can be used to compute the functions β,γ needed in different characterizations of ISS.

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR ISS$_{AG}$, ISS$_{+}$, AND ISS$_{\max}$

The main results of this section are necessary and sufficient conditions for ISS$_{AG}$, ISS$_{+}$, and ISS$_{\max}$. The results do not require a Lyapunov function but rather use the value function $V^{\delta}(x,k)$ to generate the gain functions directly. More importantly, we show that the computed functions are minimal. This type of results is not possible to obtain via Lyapunov techniques since they involve a certain conservatism in estimating the gains.

Using $V^{\delta}(x,k)$ we introduce

$$V_a^{\delta}(x):=\limsup_{k\to+\infty}V^{\delta}(x,k), \tag{9}$$

$$\gamma_{\infty}(\delta):=\sup_{x\in\mathbf{R}^n}V_a^{\delta}(x), \tag{10}$$

and

$$\gamma_a(\delta):=\max\{\gamma_{\infty}(\delta),\sup_{k\geq 0}V^{\delta}(0,k)\}. \tag{11}$$

Denote

$$\beta_a(s,k):=\sup_{|x|\leq s}V^{0}(x,k)\ . \tag{12}$$

Lemma 4.1: If the system (1) is ISS$_{+}$ with (β_+,γ_+), then $\gamma_a\in\bar{\mathcal{K}},\beta_a\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ and

$$\begin{aligned}\gamma_a(\delta)\ &\leq\ \gamma_+(\delta), &&\forall\delta\geq 0\\ \beta_a(s,k)\ &\leq\ \beta_+(s,k), &&\forall s\geq 0,\forall k\in\mathbf{Z}_+.\end{aligned}$$

Lemma 4.2: If the system (1) is ISS$_{\max}$ with $(\beta_{\max},\gamma_{\max})$, then $\gamma_a\in\bar{\mathcal{K}},\beta_a\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ and

$$\begin{aligned}\gamma_a(\delta)\ &\leq\ \gamma_{\max}(\delta), &&\forall\delta\geq 0\\ \beta_a(s,k)\ &\leq\ \beta_{\max}(s,k), &&\forall s\geq 0,\forall k\in\mathbf{Z}_+\ .\end{aligned}$$

A. Minimal β_{AG} and minimal γ_{AG}

Theorem 4.3: If the system (1) is ISS$_{AG}$ with (β_{AG},γ_{AG}) then $\gamma_{\infty}\in\bar{\mathcal{K}}$, $\beta_a\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ and

$$\begin{aligned}\gamma_{\infty}(s)\ &\leq\ \gamma_{AG}(s), &&\forall s\geq 0\\ \beta_a(s,k)\ &\leq\ \beta_{AG}(s,k), &&\forall s\geq 0,\forall k\in\mathbf{Z}_+\ .\end{aligned}$$

If, on the other hand, $\gamma_{\infty}\in\bar{\mathcal{K}}$ and $\beta_a\in\bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system (1) is ISS$_{AG}$ with $(\beta_a,\gamma_{\infty})$.

Remark 4.4: System (1) is AG if and only if $\gamma_{\infty}\in\bar{\mathcal{K}}$; System (1) is 0-GAS if and only if $\beta_a\in\bar{\mathcal{K}}\bar{\mathcal{L}}$.

B. Minimal β_+ for fixed γ_+

For $\gamma_+\in\bar{\mathcal{K}}$, we define

$$\beta^{\gamma_+}(\delta,s,k):=\max\left\{\sup_{|x|\leq s}V^{\delta}(x,k)-\gamma_+(\delta),0\right\}$$

and

$$\beta_a^{\gamma_+}(s,k):=\sup_{\delta\geq 0}\beta^{\gamma_+}(\delta,s,k). \tag{13}$$

Theorem 4.5: For fixed $\gamma_+\in\bar{\mathcal{K}}$, if there exists $\beta_+\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS$_{+}$ with (β_+,γ_+), then $\beta_a^{\gamma_+}\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ and

$$\beta_a^{\gamma_+}(s,k)\leq\beta_+(s,k),\qquad\forall s\geq 0,k\in\mathbf{Z}_+\ . \tag{14}$$

Conversely, if $\beta_a^{\gamma_+}\in\bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system (1) is ISS$_{+}$ with $(\beta_a^{\gamma_+},\gamma_+)$.

C. Minimal γ_+ for fixed β_+

For $\beta_+\in\bar{\mathcal{K}}\bar{\mathcal{L}}$, define

$$\gamma_a^{\beta_+}(\delta):=\sup_{x\in\mathbf{R}^n}\sup_{k\in\mathbf{Z}_+}\max\left\{V^{\delta}(x,k)-\beta_+(|x|,k),0\right\}. \tag{15}$$

Theorem 4.6: For fixed $\beta_+\in\bar{\mathcal{K}}\bar{\mathcal{L}}$, if there exists $\gamma_+\in\bar{\mathcal{K}}$ such that system (1) is ISS$_{+}$ with (β_+,γ_+), then $\gamma_a^{\beta_+}\in\bar{\mathcal{K}}$ and

$$\gamma_a^{\beta_+}(\delta)\leq\gamma_+(\delta),\qquad\forall\delta\geq 0\ . \tag{16}$$

Conversely, if $\gamma_a^{\beta_+}\in\bar{\mathcal{K}}$, then system (1) is ISS$_{+}$ with $(\beta_+,\gamma_a^{\beta_+})$.

D. Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$

For $\gamma_{\max}\in\bar{\mathcal{K}}$, we define

$$\tilde{\beta}^{\gamma_{\max}}(\delta,s,k):=\begin{cases}\sup_{|x|\leq s}V^{\delta}(x,k) & \text{if }\sup_{|x|\leq s}V^{\delta}(x,k)>\gamma_{\max}(\delta),\\ 0 & \text{if }\sup_{|x|\leq s}V^{\delta}(x,k)\leq\gamma_{\max}(\delta).\end{cases}$$

and

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k):=\sup_{\delta\geq 0}\tilde{\beta}^{\gamma_{\max}}(\delta,s,k). \tag{17}$$

Theorem 4.7: For a fixed $\gamma_{\max}\in\bar{\mathcal{K}}$, if there exists $\beta_{\max}\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max},\gamma_{\max})$, then $\tilde{\beta}_a^{\gamma_{\max}}\in\bar{\mathcal{K}}\bar{\mathcal{L}}$ and

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k)\leq\beta_{\max}(s,k),\qquad\forall s\geq 0,k\in\mathbf{Z}_+\ .$$

Conversely, if $\tilde{\beta}_a^{\gamma_{\max}}\in\bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system is ISS$_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}},\gamma_{\max})$.

E. Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$

For $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, define

$$\tilde{\gamma}^{\beta_{\max}}(\delta, s, k) := \begin{cases} \sup_{|x|\le s} V^\delta(x,k) & \text{if } \sup_{|x|\le s} V^\delta(x,k) > \beta_{\max}(s,k), \\ 0 & \text{if } \sup_{|x|\le s} V^\delta(x,k) \le \beta_{\max}(s,k). \end{cases}$$

and

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) := \sup_{s\ge 0} \sup_{k\in \mathbf{Z}_+} \tilde{\gamma}^{\beta_{\max}}(\delta, s, k). \qquad (18)$$

Theorem 4.8: For a fixed $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, if there exists $\gamma_{\max} \in \bar{\mathcal{K}}$ such that the system (1) is $\mathrm{ISS}_{\max}$ with $(\beta_{\max}, \gamma_{\max})$ for some $\gamma_{\max} \in \bar{\mathcal{K}}$, then $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$ and

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \le \gamma_{\max}(\delta), \qquad \forall \delta \ge 0.$$

Conversely, if $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$, then the system is $\mathrm{ISS}_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$.

Remark 4.9: It is possible to analyse several other ISS like properties using similar techniques. For example, input-to-output stability (IOS) and incremental input-to-state stability (Δ-ISS) that were respectively considered in [17], [18] and [1]. See [7] for details.

V. EXAMPLES

In this section, we present three examples to which the results of Sections III and IV are applied.

A. Example 1: ISS system with no continuous minimal asymptotic gain

Consider one dimensional system

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{2}a(|u_k|)x_k\phi(|x_k|) \qquad (19)$$

where $a(s) = 0$ for $s \in [0,9], a(s) = 1$ for $s \ge 10$ and $a(s)$ is linear on $[9,10]$; $\phi(s) = 1$ for $s \in [0,20], \phi(s) = 0$ for $s \ge 21$ and $\phi(s)$ is linear on $[20,21]$.

By computation, we have

$$\begin{aligned} \gamma_\infty(s) &\le 21, \quad \forall s \ge 0. \\ \gamma_\infty(s) &= 0, \quad \forall s \in [0,10). \\ \gamma_\infty(10) &\ge 20. \end{aligned}$$

i.e. $\gamma_\infty(s)$ has a jump point at $s = 10$; it is not a continuous function. The γ_∞ calculated by the dynamic programming method in Section IV is given by Figure 1. Since we can only calculate the values on finite points, it looks like that the function is continuous from Figure 1 (drew by MATLAB). But the jump at point $s = 10$ is clear.

Remark 5.1: System (19) is ISS_+ for $\beta(s,k) = s\frac{1}{2^k}$ and any $\mathcal{K}$-function with $\gamma(9) \ge 21$. However, system

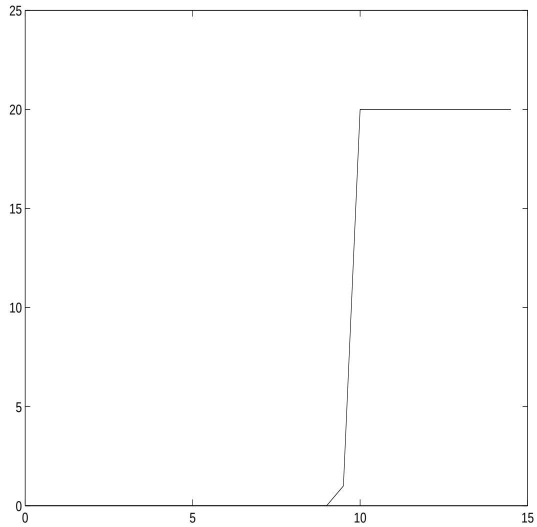

Fig. 1. $\gamma_\infty(s)$ obtained by dynamic programming

(19) can not be ISS_+ for γ_∞ and any $\beta(s,k) \in \bar{\mathcal{K}}\bar{\mathcal{L}}$! The reason is that it is impossible to find a $\beta(s,k) \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that $\beta(s,k) \ge (1 - \frac{1}{2}\delta)^k s$ for all $1 > \delta > 0$. In another word, this example shows that γ_∞ is not a good candidate of γ_+.

B. Example 2: A class of scalar linear systems

Consider the class of scalar linear systems given by

$$x_{k+1} = ax_k + bu_k, \qquad (20)$$

where $0 < a < 1$ and $b \ge 0$. By direct calculation, DPE (7) and initialisation (8) for V^δ imply that for any $k \ge 0$,

$$V^\delta(x,k) = a^k|x| + \left(\frac{1-a^k}{1-a}\right)b\delta. \qquad (21)$$

ISS_{AG} property: Applying definitions (9), (10), and (12) of respectively $V_a^\delta(x)$, $\gamma_\infty(\delta)$ and $\beta_a(s,k)$,

$$\begin{aligned} V_a^\delta(x) &= \limsup_{k\to\infty} V^\delta(x,k) = \left(\frac{b}{1-a}\right)\delta, \\ \gamma_\infty(\delta) &= \sup_{x\in\mathbf{R}^n} V_a^\delta(x) = \left(\frac{b}{1-a}\right)\delta, \\ \beta_a(s,k) &= \sup_{|x|\le s} V^0(x,k) = sa^k. \end{aligned}$$

Since $\gamma_\infty \in \bar{\mathcal{K}}$ and $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, Theorem 4.3 implies that system (20) is ISS_{AG} with (β_a, γ_∞).

ISS_+ property: Applying definition (11) of $\gamma_a(\delta)$,

$$\begin{aligned} \gamma_a(\delta) &= \max\left\{\gamma_\infty(\delta), \sup_{k\ge 0} V^\delta(0,k)\right\} \qquad (22) \\ &= \max\left\{\gamma_\infty(\delta), \left(\frac{b}{1-a}\right)\delta\right\} = \gamma_\infty(\delta). \end{aligned}$$

(i) Minimal β_+ for fixed γ_+: Using γ_a as a candidate (fixed) gain in testing ISS$_+$ (i.e. $\gamma_+ = \gamma_a$), we obtain the minimal ISS$_+$ transient bound for gain γ_+,

$$\beta_a^{\gamma_+}(s,k) \; = \; \sup_{\delta \geq 0} \beta^{\gamma_+}(\delta, s, k) \; = \; s a^k = \beta_a(s,k),$$

Theorem 4.5 then implies that system (20) is ISS$_+$ with $(\beta_a^{\gamma_+}, \gamma_+)$, where $\gamma_+ = \gamma_a$ and $\beta_a^{\gamma_+}$ is the minimal corresponding transient bound.

(ii) Minimal γ_+ for fixed β_+: Using β_a as a candidate (fixed) transient bound in testing ISS$_+$ (i.e. $\beta_+ = \beta_a$), we have

$$\gamma_a^{\beta_+}(\delta) \; = \; \left(\frac{b}{1-a} \right) \delta \; = \; \gamma_a(\delta).$$

Theorem 4.6 implies that system (20) is ISS$_+$ with $(\beta_+, \gamma_a^{\beta_+})$, where $\beta_+ = \beta_a$ and $\gamma_a^{\beta_+}$ is the minimal corresponding gain.

ISS$_{\max}$ property:

(i) Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$: Using γ_a as a candidate (fixed) gain in testing ISS$_{\max}$ (i.e. $\gamma_{\max} = \gamma_a$), we have

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) \; = \; s,$$

which is not of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$. Hence, the gain γ_a is too small to be a gain candidate for computing the minimal transient bound. To illustrate this point further, suppose a slightly larger candidate gain is chosen, namely

$$\gamma_{\max}(\delta) \; = \; (1+\varepsilon)\gamma_a(\delta)$$

where $\varepsilon > 0$ is fixed and small, then

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) \; = \; \left(\frac{1+\varepsilon}{a^k + \varepsilon} \right) s a^k \; = \; \left(\frac{1+\varepsilon}{a^k+\varepsilon} \right) \beta_a(s,k),$$

which is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ for any $\varepsilon > 0$. Hence, by Theorem 4.7, system (20) is ISS$_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$, $\gamma_{\max} = (1+\varepsilon)\gamma_a$.

(ii) Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$: Using β_a as a candidate (fixed) transient bound in testing ISS$_{\max}$ (i.e. $\beta_{\max} = \beta_a$), we obtain

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \; \geq \; \sup_{s \geq 0} s \; = \; \infty, \tag{23}$$

for all $\delta > 0$, which is clearly not of class $\bar{\mathcal{K}}$. As in the minimal transient bound case, this implies that the transient bound β_a is too small to be a candidate transient bound for ISS$_{\max}$. To illustrate that this system is ISS$_{\max}$, choose the slightly larger transient bound

$$\beta_{\max}(s,k) \; = \; (1+\varepsilon)\beta_a(s,k)$$

where $\varepsilon > 0$, then

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \; = \; \left(1 + \frac{1}{\varepsilon}\right) \frac{b\delta}{1-a} \; = \; \left(1 + \frac{1}{\varepsilon}\right) \gamma_a(\delta),$$

which is of class $\bar{\mathcal{K}}$. Theorem 4.8 then implies that system (20) is ISS$_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$, $\beta_{\max} = (1+\varepsilon)\beta_a$.

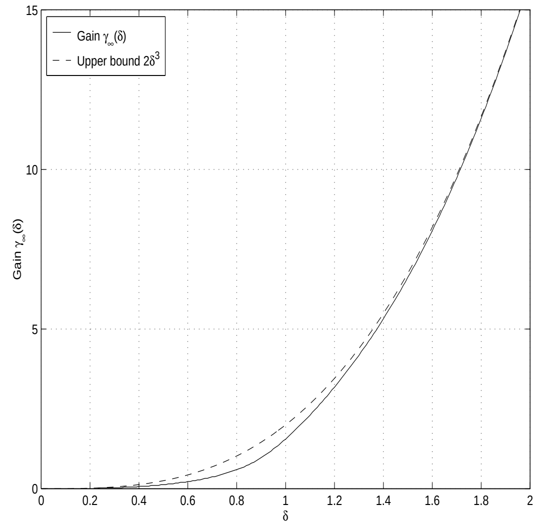

Fig. 2. $\gamma_\infty(\delta)$ for system (24).

C. Example 3: A scalar nonlinear system

Consider the scalar nonlinear system

$$x_{k+1} \; = \; \frac{x_k^3}{2(1+x_k^2)} + u_k^3 \tag{24}$$

A standard numerical scheme is applied to DPE (7) using the following discretized input bound space Δ, state space X and control space U:

$$\begin{aligned} \Delta \; &= \; \{\delta \in \mathbf{R} : 0 \leq \delta \leq 3\}_{N_\Delta = 301}, \\ X \; &= \; \{x \in \mathbf{R} : |x| \leq 15\}_{N_X = 1501}, \\ U^\delta \; &= \; \{u \in \mathbf{R} : |u| \leq \delta\}_{N_U = 201}, \;\; \delta \in \Delta. \end{aligned}$$

Here, N_Δ, N_X and N_U respectively refer to the number of points in each of the discretized spaces Δ, X and U^δ. The result of the applying DPE (7) over these discretized spaces is an approximation for V^δ. With $V^\delta(x,k)$ computed for all $\delta \in \Delta$, computation of approximations for the remaining quantities is then possible.

ISS$_{AG}$ property: Computing $V_a^\delta(x)$ from definition (9) yields (in this case) functions that are independent of x. γ_∞ then follows from definition (10). Note that since $\left| \frac{x_k^3}{2(1+x_k^2)} \right| \leq \left| \frac{x_k}{2} \right|$, it is easy to prove that $\gamma_\infty(\delta) \leq 2\delta^3$, thereby providing a useful upper bound for this gain. β_a likewise follows from definition (12). The resulting approximations are illustrated in Figures 2 and 3 respectively. As $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_\infty \in \bar{\mathcal{K}}$ (at least over the discretized spaces used in the computation), Theorem 4.3 implies that system (24) is ISS$_{AG}$ with (β_a, γ_∞).

The results about the ISS$_+$ property and the ISS$_{\max}$ property are omitted due to the space limitation.

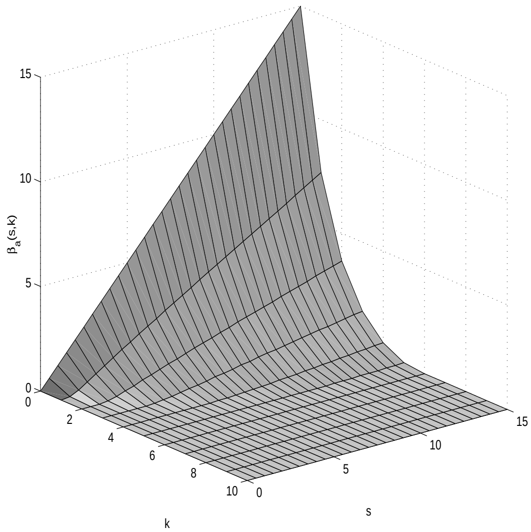

Fig. 3. $\beta_a(s,k)$ for system (24).

VI. CONCLUSIONS

We have presented results for verifying different characterizations of ISS via dynamic programming. Formulas for minimum nonlinear gains and bounds on transients for different characterizations are presented. We illustrated our approach by three examples.

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