The Nonlinearity Structure of Point Feature SLAM Problems with Spherical Covariance Matrices

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Abstract

This paper proves that the optimization problem of one-step point feature Simultaneous Localization and Mapping (SLAM) is equivalent to a nonlinear optimization problem of a single variable when the associated uncertainties can be described using spherical covariance matrices. Furthermore, it is proven that this optimization problem has at most two minima. The necessary and sufficient conditions for the existence of one or two minima are derived in a form that can be easily evaluated using observation and odometry data. It is demonstrated that more than one minimum exists only when the observation and odometry data are extremely inconsistent with each other. A numerical algorithm based on bisection is proposed for solving the one dimensional nonlinear optimization problem. It is shown that the approach extends to joining of two maps, thus can be used to obtain an approximate solution to the complete SLAM problem through map joining.

Key words: SLAM; least squares; one-dimension optimization problem; minima analysis.

1 Introduction

Simultaneous Localization and Mapping (SLAM) is the problem of building a map of the environment using a mobile robot and at the same time estimating the location of the robot within the map (Dissanayake, Newman, Clark, Durrant-Whyte & Csorba 2001)(Bailey & Durrant-Whyte 2006). SLAM is a fundamental problem for mobile robot navigation and has been investigated by robotic researchers for more than ten years (Tardos, Neira, Newman & Leonard 2002)(Durrant-Whyte & Bailey 2006)(Frese 2006)(Huang & Dissanayake 2007)(Kummerle, Steder, Dornhege, Ruhnke, Grisetti, Stachniss & Kleiner 2009)(Cadena & Neira 2010).

Although many efficient SLAM algorithms have been developed, and the sparseness of the information matrix in different SLAM formulations is now well understood and exploited thoroughly (e.g. (Agrawal & Konolige 2008)(Dellaert & Kaess 2006)(Huang, Wang & Dissanayake 2008)), the underlying structure of the nonlinearity in SLAM has not been fully understood as yet. For example, Olson, Leonard & Teller (2006) demonstrates that pose graph SLAM, formulated as a high dimensional nonlinear optimization problem, can be solved using stochastic gradient descent (SGD) by simply dealing with each of the constraints individually. Recently, a more efficient SLAM algorithm, the so-called tree-based network optimizer (TORO), was proposed in Grisetti, Rizzini, Stachniss, Olson & Burgard (2008) where a tree structure is used on top of the SGD approach. Surprisingly, very large scale SLAM problems can be very efficiently solved in this way without the need for good initial values, especially when the covariance matrices of the relative poses are close to spherical (Grisetti, Stachniss & Burgard 2009). Some initial attempts to obtain a closed-form solution to pose graph SLAM have been made in Rizzini (2009), and in Martinez, Morales, Mandow & Garcia-Cerezo (2010) and Carlone, Aragues, Castellanos & Bona (2011), a linear approximation approach for pose graph SLAM is proposed.

For point feature based SLAM problem, our initial investigation has also shown some interesting phenomenon when a simple Gauss-Newton algorithm is applied to solve the SLAM as an optimization problem (Huang, Lai, Frese &
Dissanayake 2010). For the Victoria Park dataset (Guivant & Nebot 2001), the algorithm can converge to the globally optimal solution with random initial values 80% of the time\(^1\). However, this “magic” convergence happens only when the covariances of observations and odometries are set to be spherical matrices but not when the original covariance matrices are used.

Although very high dimensional nonlinear optimization problems typically have many local minima, the above convergence results indicate that this may not be the case for SLAM problems. This motivates us to analyze the structure of the nonlinearity and the number of local minima involved in SLAM. The main contribution of this paper is the theoretical insight into the nonlinearity structure of SLAM problems and a demonstration that one-step SLAM problem is equivalent to a non-linear optimization problem in a single variable. Furthermore, a numerical algorithm based on bisection is proposed to solve the one-dimensional optimization problem. It is also shown that this algorithm can be used for joining two local maps by approximating the local map covariance matrices with spherical ones, such that an approximate solution to the complete SLAM problem through sequential map joining can be obtained. This, however, is meant as an application of our theoretical results, not as a full new SLAM algorithm. This paper is based on our preliminary work on feature based SLAM (Huang, Wang, Frese & Dissanayake 2012) and pose graph SLAM (Wang, Hu, Huang & Dissanayake 2012). The key improvements in this paper over Huang et al. (2012) and Wang et al. (2012) include: (1) a new definition of spherical covariance matrices, a new compact way to express the high dimensional optimization problems, and a new and more intuitive way to prove that the high dimensional problem is equivalent to a one dimensional problem; (2) the wrap of 2π periodicity of angles is taken into account when analyzing the optimization problem, which is more practical; (3) a bisection algorithm for solving the one dimensional problem for feature based SLAM is given; and (4) more intuitive explanations of the results are presented.

The paper is organized as follows. Section 2 states the least squares SLAM formulation. In Section 3, the definition of spherical matrices is introduced, and an alternative SLAM formulation is derived when covariance matrices are spherical. In Section 4, it is shown that the one-step SLAM problem is equivalent to a nonlinear optimization problem in a single variable. Section 5 provides an analysis of the number of minima for one-step SLAM. In Section 6, a numerical algorithm based on bisection is proposed to obtain the optimal solution of SLAM. In Section 7, experiments are performed to validate the analysis and the proposed numerical algorithm. Section 8 concludes the paper and the appendix presents the proof of a central, but more technical theorem related to the minima analysis.

\(^1\) Here the preprocessed data is used where data association is provided. The preprocessed data is available on OpenSLAM website: http://openslam.org/ under project 2D I-SLSJF.

**Notations:** Throughout the paper, the superscript \(T\) and \(-1\) stand for the transposition and the inverse of a matrix, respectively; \(C > 0\) means that matrix \(C\) is real symmetric and positive definite; \(I\) and \(I_n\) represent the identity matrix with appropriate dimension and the identity matrix with dimension \(n\), respectively; \(0\) represents the zero matrix with a compatible dimension; and

\[
|e|^2_C = e^T C^{-1} e
\]

denotes the weighted norm of vector \(e\) with a given covariance matrix \(C\). \(\text{wrap}(\theta)\) is a function that converts an arbitrary angle into the equivalent angle in the range \([-\pi, \pi)\), e.g., if \(\theta = \frac{3\pi}{2}\), then \(\text{wrap}(\theta) = -\frac{\pi}{2}\). The symbol \(\text{diag}(A_1, \ldots, A_n)\) denotes a diagonal matrix whose elements are \(A_1, \ldots, A_n\).

## 2 Least squares SLAM problem

### 2.1 Point feature based SLAM problem

Input to the SLAM problem contains two kinds of information. Odometry provides the geometric relationship between two consecutive poses. Observations provide the relative position of observed features with respect to the pose from which the observations are made. The full least squares SLAM formulation (Dellaert & Kaess 2006) uses the odometry and observation information to estimate all the robot poses and all the feature positions.

In this paper, we use \(Z_i^j \in \mathbb{R}^{2 \times 1}\) to denote the observation made from pose \(r_i\) to feature \(f_j\) (observed \(x\) and \(y\) coordinates of the feature). We use \(O_{i-1}^i \in \mathbb{R}^{3 \times 1}\) (1 \(\leq i \leq p\)) to denote the odometry information between pose \(r_{i-1}\) and pose \(r_i\) (relative \(x, y\) position and orientation angle), \(P_{Z_i} \in \mathbb{R}^{2 \times 2} > 0\) and \(P_{O_{i}} \in \mathbb{R}^{3\times 3} > 0\) are the corresponding covariance matrices of the observation and odometry noises. Here the noises are assumed to be zero-mean Gaussian. Fig. 1 presents a simple SLAM problem, where there are 4 odometries and 5 observations.

Fig. 1. The SLAM problem with 5 poses, 3 features and 5 observations, the initial pose \(r_0\) is fixed to \((0,0,0)\).
2.2 Least squares SLAM formulation

Suppose a number of 2D point features $f_1, f_2, \ldots, f_n$ in the environments are observed from a sequence of 2D robot poses $r_0, r_1, r_2, \ldots, r_p$. The first robot pose (pose $r_0$) is chosen as the origin of the global coordinate frame.

We use $X_{f_j} = [x_{f_j}, y_{f_j}]^T$ to denote the $x, y$ position of feature $f_j$. $X_{r_i} = [x_{r_i}, y_{r_i}]^T$ denotes the $x, y$ position of robot pose $r_i$ while $\phi_{r_i}$ denotes the orientation of pose $r_i$. $R_{r_i}$ is the rotation matrix of pose $r_i$ given by

$$R_{r_i} = R(\phi_{r_i}) = \begin{bmatrix} \cos \phi_{r_i} & -\sin \phi_{r_i} \\ \sin \phi_{r_i} & \cos \phi_{r_i} \end{bmatrix}$$ \hspace{1cm} (2)

The least squares SLAM formulation uses the odometry and observation information to estimate the state vector containing all the robot poses and all the feature positions

$$X = [X^T_{f_1}, \ldots, X^T_{f_n}, X^T_{r_1}, \phi_{r_1}, \ldots, X^T_{r_n}, \phi_{r_n}]^T$$

and the SLAM problem is to minimize

$$F(X) = \sum_{i=1}^{p} |O^{-1}_{r_i} - H^{O_i}(X)|^2_{P_{O_i}} + \sum_{i,j} |Z^i_{r_j} - H^{Z_i}(X)|^2_{P_{Z_j}}$$ \hspace{1cm} (3)

Note here that we have used the notation in (1) to simplify the expression.

In (3), $H^{O_i}(X)$ and $H^{Z_i}(X)$ are the corresponding functions relating $O_{r_i}^{-1}$ and $Z_{r_j}$ to the state $X$. The odometry is a function of two poses $[X^T_{r_{i-1}}, \phi_{r_{i-1}}]^T$ and $[X^T_{r_i}, \phi_{r_i}]^T$ and is given by

$$H^{O_i}(X) = \begin{bmatrix} R_{r_{i-1}}^T(X_{r_i} - X_{r_{i-1}}) \\ \phi_{r_i} - \phi_{r_{i-1}} \end{bmatrix}$$

For example, in Fig. 1, the function of odometry between $r_1$ and $r_2$ is

$$H^{O_2}(X) = \begin{bmatrix} R_{r_1}^T(X_{r_2} - X_{r_1}) \\ \phi_{r_2} - \phi_{r_1} \end{bmatrix} = \begin{bmatrix} (x_{r_2} - x_{r_1}) \cos(\phi_{r_1}) + (y_{r_2} - y_{r_1}) \sin(\phi_{r_1}) \\ -(x_{r_2} - x_{r_1}) \sin(\phi_{r_1}) + (y_{r_2} - y_{r_1}) \cos(\phi_{r_1}) \end{bmatrix}$$

The observation is a function of one pose $(X^T_{r_i}, \phi_{r_i})^T$ and one feature position $X_{f_j}$ which is given by

$$H^{Z_i}(X) = R_{r_i}^T(X_{f_j} - X_{r_i})$$

In Fig. 1, the function of the observation of $f_2$ by $r_1$ is

$$H^{Z_2}(X) = R_{r_1}^T(X_{f_2} - X_{r_1}) = \begin{bmatrix} (x_{f_2} - x_{r_1}) \cos(\phi_{r_1}) + (y_{f_2} - y_{r_1}) \sin(\phi_{r_1}) \\ -(x_{f_2} - x_{r_1}) \sin(\phi_{r_1}) + (y_{f_2} - y_{r_1}) \cos(\phi_{r_1}) \end{bmatrix}$$

In particular, since $\phi_{r_0} = 0$ and $X_{r_0} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, the odometry function from robot $r_0$ to $r_1$ is given by

$$H^{O_1}(X) = \begin{bmatrix} X_{r_1} \\ \phi_{r_1} \end{bmatrix}$$

and the observation function from robot $r_0$ to $f_j$ is given by

$$H^{Z_j}(X) = X_{f_j}$$

“One-step SLAM” problem is a special case of (3) where the number of robot poses is two, i.e. $p = 1$.

3 Alternative formulation with spherical covariances

Note that $H^{O_i}(X)$ and $H^{Z_i}(X)$ in (3) are all nonlinear functions. In this section, we will show that there is an alternative formulation of $F(X)$ which is much simpler if the covariance matrices $P_{Z_j}$ and $P_{O_i}$ are all spherical.

3.1 Definition of spherical matrices

We first introduce the definition of spherical matrices.

**Definition 1** A matrix $A \in \mathbb{R}^{2 \times 2}$ is called spherical if it commutes with $R(\phi)$ in (2) for every $\phi$, i.e. $AR(\phi) = R(\phi)A$ for every $\phi$. A matrix $B \in \mathbb{R}^{3 \times 3}$ is called spherical if it has the format of $B = \text{diag}(A, a)$ where $A \in \mathbb{R}^{2 \times 2}$ is spherical and $a$ is a real number.

Intuitively, such a spherical matrix treats every direction in the plane the same way, so it does not make a difference, whether a vector is rotated before or after application of the matrix.

**Remark 1** A $2 \times 2$ positive definite spherical matrix has the format $A = \text{diag}(a, a)$ with $a > 0$. A $3 \times 3$ positive definite spherical matrix has the format $A = \text{diag}(a, a, b)$ with $a > 0$ and $b > 0$.

We will also introduce a more general definition of spherical matrices which will be used in the following sections of this paper.

**Definition 2** Let $R_{2n}(\phi) \in \mathbb{R}^{2n \times 2n}$ be a block-diagonal matrix with $\phi$-rotation matrices i.e., $R(\phi)$, in every $2 \times 2$
block on the diagonal. Matrix $A \in \mathbb{R}^{2m \times 2n}$ is called spherical if it commutes with in-plane rotation, i.e. $AR_{2m} (\phi) = R_{2m} (\phi)A$ for every $\phi$.

As an example, for $m = 3, n = 2$,

$$A \in \mathbb{R}^{6 \times 4} = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \\ I_2 & -I_2 \end{bmatrix}$$

is a spherical matrix, since we have $AR_{4} (\phi) = R_{4} (\phi)A$ for every $\phi$ where $R_{4} (\phi) = \text{diag}(R(\phi), R(\phi), R(\phi), R(\phi))$.

Definition 2 makes sense for matrices operating on stacked 2D vectors with alternating $x$ and $y$ coordinates. Intuitively, these matrices treat all directions the same, it does not make a difference whether all 2D vectors in a stacked vector rotated before or after the application of the matrix. Note that all 2D components must be rotated by the same angle. A non-trivial example is the matrix $\begin{bmatrix} I_2 & \frac{1}{3} I_2 & \frac{1}{3} I_2 \end{bmatrix}$, which maps 3 points to their center. It makes no difference whether the points are rotated or the resulting center is rotated.

**Remark 2** The sum, difference, product, inverse, transpose of spherical matrices are also spherical matrices. If $P > 0$ is spherical and $R$ is a rotation matrix, then for any $x$, $Rx$ and $x$ have the same weighted norm as shown below:

$$|Rx|_P^2 = x^T R^T P^{-1} Rx = x^T R^T P R P^{-1} x = x^T P^{-1} x = |x|_P^2$$

where $P^{-1}$ is also spherical.

### 3.2 Alternative formulation

From Remark 2, when the covariance $P_{Z_j}$ is a $2 \times 2$ spherical positive definite matrix, we have

$$|Z_j^i - H Z_j^i (X)|^2_{P_{Z_j}} = |Z_j^i - R_{xy}^T (X_{ij} - X_{ri})|^2_{P_{Z_j}}$$

$$= |R_{xy} Z_j^i - X_{ij} - X_{ri}|^2_{P_{Z_j}}$$

(4)

Similarly, when the covariance $P_{O_i}$ is a $3 \times 3$ spherical positive definite matrix with the format

$$P_{O_i} = \text{diag}(P_{O_i^{x'y}}, P_{O_i^{z'}})$$

where $P_{O_i^{x'y}}$ is a $2 \times 2$ spherical positive definite matrix, the odometry term

$$|O_i^{z'} - H O_i (X)|^2_{P_{O_i}}$$

with $O_i^{z'} = \begin{bmatrix} O_i^{z'-1} \\ O_i^{z'-1} \end{bmatrix}$ becomes

$$|O_i^{z'-1} - R_{xy}^T (X_{ri} - X_{ri-1})|^2_{P_{O_i^{x'y}}}$$

$$+ P_{O_i}^{-1} [\text{wrap}(O_i^{z'-1} - \phi_{ri} + \phi_{ri-1})]^2$$

$$= |R_{xy} O_i^{z'-1} (X_{ri} - X_{ri-1})|^2_{P_{O_i^{x'y}}}$$

$$+ P_{O_i}^{-1} [\text{wrap}(O_i^{z'-1} - \phi_{ri} + \phi_{ri-1})]^2$$

Note that the difference between any two angles is within $[-\pi, \pi)$, so we wrap the error in orientations.

Now the objective function in least squares SLAM formulation can be written as

$$F(X) = \sum_{i,j} |R_{rij} Z_j^i - (X_{f_j} - X_{r_i})|^2_{P_{Z_j}}$$

$$+ \sum_{i=1}^p |R_{ri-1} O_i^{z'-1} - (X_{r_i} - X_{r_{i-1}})|^2_{P_{O_i^{x'y}}}$$

$$+ \sum_{i=1}^p P_{O_i}^{-1} [\text{wrap}(O_i^{z'-1} - \phi_{ri} + \phi_{ri-1})]^2$$

where $\phi_{r_0} = 0, R_{r_0} = I, X_{r_0} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

### 4 One-step SLAM problem

As pointed out in Cadena & Neira (2010) and Huang, Wang \\ & Dissanayake (2008), all SLAM problems can be decomposed to that of joining two maps, which is similar to a one-step SLAM problem. To understand the nonlinearity of the general SLAM problem, it is essential to analyze the simple case of one-step $n$-feature SLAM which is shown in Fig. 2.

![Fig. 2. One-step SLAM problem with $n$ features](image)

4.1 A compact form of the objective function

For one-step SLAM with spherical covariance matrices, assume the $n$ features are observed from both pose $r_0$ and pose $r_1$, then the objective function in least squares SLAM
formulation is
\[ F(X) = \sum_{j=1}^{n} |Z_j^0 - X_{f_j}p_{z_j^0}|^2 + \sum_{j=1}^{n} |R_{r_j}Z_j^0 - (X_{f_j} - X_{r_j})|^2 p_{z_j^0} + |O_{x_{w_j}}^0 - X_{r_j}p_{z_{x_{w_j}}^0} + p_{z_{x_{w_j}}^0}^{-1}(\text{wrap}(O_{x_{w_j}}^0 - \phi_{r_j}))|^2 \]

To simplify the notations, we use \( \phi \) to denote \( \phi_{r_j} \), \( z_\phi \) to denote \( O_{x_{w_j}}^0 \cdot p_{\phi} \) to denote \( p_{O_{x_{w_j}}} \). Then the objective function of the spherical one-step SLAM problem is
\[ F(X) = \sum_{j=1}^{n} |Z_j^0 - X_{f_j}p_{z_j^0}|^2 + \sum_{j=1}^{n} |R_{r_j}Z_j^0 - (X_{f_j} - X_{r_j})|^2 p_{z_j^0} + |O_{x_{w_j}}^0 - X_{r_j}p_{z_{x_{w_j}}^0} + p_{z_{x_{w_j}}^0}^{-1}(\text{wrap}(z_\phi - \phi))^2 \]

This function can be further written in a compact form as
\[ F_1(X_L, \phi) = |AX_L - z_0 - \hat{R}(\phi)z_1|^2 + p_{z_\phi}^{-1}(\text{wrap}(z_\phi - \phi))^2 \]

where \( \hat{R}(\phi) \) is defined in Definition 2 (for brevity, the dimension subscripts of \( \hat{R}(\phi) \) will be omitted later), and

\[ A = \begin{bmatrix} I_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_2 & 0 \\ 0 & \cdots & 0 & I_2 \\ I_2 & \cdots & 0 & -I_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_2 & -I_2 \end{bmatrix}, \quad X_L = \begin{bmatrix} X_{f_1} \\ \vdots \\ X_{r_1} \end{bmatrix}, \quad z_0 = \begin{bmatrix} Z_1^0 \\ \vdots \\ Z_n^0 \\ 0 \end{bmatrix} \]

\[ C = \text{diag}(P_{z_1}, \ldots, P_{z_n}, P_{O_{x_{w_1}}}, P_{O_{x_{w_2}}}, \ldots, P_{O_{x_{w_n}}}) \]

Remark 3 In this new compact form, \( X_L \) represents the “linear” part of the state vector \( X \); \( z_0 \) contains measurements taken from pose \( r_0 \) and the translation part of the odometry; \( z_1 \) contains measurements taken from pose \( r_1 \), which are then rotated according to the orientation of pose \( r_1 \); \( C \) is a block-diagonal matrix made from the \( 2 \times 2 \) covariance matrices.

Definition 3 In this paper, the function
\[ F_0(X_L, \phi) = |AX_L - z_0 - \hat{R}(\phi)z_1|^2 \]

is called the objective function of a spherical one-step SLAM problem without orientation link. The function (6) is called the objective function of a spherical one-step SLAM problem with an orientation link. Note that here both \( A \) and \( C \) are spherical matrices according to Definition 2.

Using this compact form of SLAM objective functions, we can easily prove that the one-step SLAM problem is equivalent to a one dimensional optimization problem. In fact, for any fixed \( \phi \), the problem is linear and the optimal \( X_L \) to minimize \( F_0(X_L, \phi) \) in (9) or minimize \( F_1(X_L, \phi) \) in (6) can be obtained in a closed-form which has a simple dependence on \( \phi \). Thus the optimization problem becomes a one dimensional optimization problem with respect to the single variable \( \phi \).

4.2 Spherical one-step SLAM

The following lemma is essential for later developments.

Lemma 1 For a linear least-squares problem \( |AX - b|^2 \), its minimum can be obtained by
\[ \min_{X} |AX - b|^2 = |b|^2 \]
where \( Q^{-1} = C^{-1} - C^{-1}A(A^TC^{-1}A)^{-1}A^TC^{-1} \), and the solution is also the global minimum.

Proof. Note that the optimal solution to \( \min_{X} |AX - b|^2 \) is
\[ X^* = (A^TC^{-1}A)^{-1}A^TC^{-1}b \]
substituting \( X^* \) into \( |AX - b|^2 \), we can get \( |b|^2 \). This completes the proof. \( \square \)

Theorem 1 For a spherical one-step \( n \)-feature SLAM problem, the optimal solution to the following optimization problem
\[ \min_{X_L, \phi} F(X_L, \phi) \]
where
\[ F(X_L, \phi) = |AX_L - z_0 - \hat{R}(\phi)z_1|^2 + \chi P_{z_\phi}^{-1}(\text{wrap}(z_\phi - \phi))^2 \]
can be obtained by firstly solving the following one dimension optimization problem
\[ \min_{\phi} f_\chi(\phi) \]
to obtain \( \phi^* \), then solving \( X_L^* \) by
\[ X_L^* = (A^TC^{-1}A)^{-1}A^TC^{-1}(z_0 + \hat{R}(\phi^*)z_1) \]
where
\[ f_\chi(\phi) = c_0 - 2a_0 \cos(\phi - \phi_0) + \chi P_{z_\phi}^{-1}(\text{wrap}(z_\phi - \phi))^2 \]
and $c_0, a_0, \phi_0$ are constants given by

$$c_0 = z_0^T Q^{-1} z_0 + z_1^T Q^{-1} z_1$$  \hspace{1cm} (12)

$$a_0 = \sqrt{(z_0^T Q^{-1} z_1)^2 + (z_0^T Q^{-1} z_1^+)^2},$$  \hspace{1cm} (13)

$$\phi_0 = \arctan(\frac{-z_0^T Q^{-1} z_1^+}{-z_0^T Q^{-1} z_1})$$  \hspace{1cm} (14)

and $z_1^+ = R(\frac{\pi}{2}) z_1$; $\chi \in \{0, 1\}$, if $\chi = 0$, there is no orientation link, else if $\chi = 1$, there is an orientation link.

Proof. Let

$$f_0(\phi) = \min_{\chi_L} F_0(X_L, \phi) = \min_{\chi_L} |AX_L - z_0 - R(\phi) z_1|^2,$$

applying Lemma 1, and observing that $Q$ is again a spherical matrix, we have

$$f_0(\phi) = |z_0 + R(\phi) z_1|^2 = (z_0 + R(\phi) z_1)^T Q^{-1} (z_0 + R(\phi) z_1)$$

$$= z_0^T Q^{-1} z_0 + 2 z_0^T Q^{-1} R(\phi) z_1 + z_1^T R(\phi) Q^{-1} R(\phi) z_1$$

$$= c_0 + 2 z_0^T Q^{-1} (z_1 \cos \phi + z_1^+) \sin \phi$$

$$= c_0 + 2 \left( z_0^T Q^{-1} z_1 \right) \cos \phi + (2 z_0^T Q^{-1} z_1^+) \sin \phi$$

$$= c_0 - 2 \left[ -z_0^T Q^{-1} z_1 \right] \cos \phi + \left[ -z_0^T Q^{-1} z_1^+ \right] \sin \phi$$

$$= c_0 - 2 a_0 \cos(\phi - \phi_0)$$

where $c_0, a_0, \phi_0$ are given by (12), (13), (14). Thus, $\phi$ can be obtained by solving optimization problem (10), then from the proof of Lemma 1, $X_0^*$ can be obtained by (11) where $\phi^*$ is the solution to (10), this completes the proof. \qed

Remark 4 It should be noted that the spherical covariance matrices play a key role in developing the results in Theorem 1. If the covariance matrix is non-spherical, the matrix $Q$ will be a function of $\phi$ and the function $f_0$ will be far more complicated. The non-trivial insight in Theorem 1 is that $f_0(\phi)$ has such a simple form, only a shifted cosine function in $\phi$.

Remark 5 The spherical one-step n-feature SLAM problem without odometry is similar to the problem of finding the relative transformation between two coordinate frames given two corresponding point sets (Arun, Huang & Blostein 1987), both of which are formulated as a problem of determining a rotation matrix. The difference between the approach proposed in this paper and that of Arun et al. (1987) is that a numerical algorithm through using singular value decomposition (SVD) is proposed to solve the 3D problem in Arun et al. (1987), while closed-form solution to the simpler 2D case is obtained in this paper, i.e., the rotation matrix $R(\phi_0)$ can be obtained from formula (2) where $\phi_0$ is given by (14).

5 Minima analysis of the spherical one-step SLAM

In this section, we analyze the minima characteristic of the objective function of one-step SLAM problem. By virtue of Theorem 1, the problem is essentially a one dimensional problem and when there is no orientation link, the one dimensional function $f_0(\phi)$ is periodic with a single minimum of $\phi_0$. With the additional orientation link (i.e., $\chi = 1$ in Theorem 1), $f_0(\phi)$ in (10) is still periodic because of the wrap function. So it is enough to perform the analysis within any interval with the length of $2\pi$. Without loss of generality, we consider interval $\phi \in [z_\phi - \pi, z_\phi + \pi)$. Within this interval, the wrap is not needed and we have the following theorem which analyzes the minima of one-step SLAM problem with an orientation link.

Theorem 2 Let $c_0 \geq 0$, $a_0 > 0$, $\phi_0 \in [-\pi, \pi)$ be constants defined in (12)-(14), $p_\phi > 0$, $z_\phi \in [-\pi, \pi)$ are given constants, then the function

$$f_1(\phi) = c_0 - 2 a_0 \cos(\phi - \phi_0) + p_\phi^{-1} (z_\phi - \phi)^2$$  \hspace{1cm} (15)

has at least one and at most two minima in the interval $\phi \in [z_\phi - \pi, z_\phi + \pi)$. Moreover, there are two minima if and only if

$$a \geq 1$$  \hspace{1cm} (16)

$$-a \sin C_\phi - \pi \leq 0$$  \hspace{1cm} (17)

$$\sqrt{a^2 - 1} + \varphi_1 \geq 0$$  \hspace{1cm} (18)

$$-\sqrt{a^2 - 1} + \varphi_2 \leq 0$$  \hspace{1cm} (19)

$$-a \sin C_\phi + \pi \geq 0$$  \hspace{1cm} (20)

$$\varphi_1 - \varphi_2 \leq 0$$  \hspace{1cm} (21)

hold simultaneously, where

$$a = a_0 p_\phi, C_\phi = z_\phi - \phi_0$$

and

$$\varphi_1 = \text{wrap}(\arccos(-\frac{1}{a}) - C_\phi),$$

$$\varphi_2 = \text{wrap}(\arccos(-\frac{1}{a}) - C_\phi).$$  \hspace{1cm} (22)

Proof. See Appendix. \qed

Remark 6 Fig. 3 illustrates conditions (16)-(21). The possible pair of $a, C_\phi$ when there are two minima is shown in the shaded area. For example, it can be seen that if $a < 1$, there is only one minimum. If $|C_\phi| < \arcsin(\frac{1}{a}) = 1.2626$, there is also only one minimum. Note that $C_\phi$ is the difference between the orientation measurement from odometry ($z_\phi$) and the orientation that best fits the observation data ($\phi_0$). So in any practical scenario, $|C_\phi|$ is smaller than 1.2626 (72.3 degrees) and there is only one minimum. There are two minima only when the odometry data and the observation data are extremely inconsistent.

6
\[ n = \text{number of features that appear only in local map } 1 \mbox{ and local map } 2, \text{ and } n_3 = \text{number of features that appear only in local map } 1 \mbox{ which is the origin of the global map.} \]

\[ r_1 \text{ is the end pose in local map } 1 \mbox{ as well as the start pose of local map } 2, \]

\[ r_2 \text{ is the end pose in local map } 2. \]

In fact, only common features are important in the map joining problem, and it is evident that the map joining problem in Fig. 4 with \( r_2 \) and \( n_1 > 0, n_2 > 0 \) is equivalent to the case without \( r_2 \) and \( n_1 = 0, n_2 = 0 \) which is shown in Fig. 2.

6 A bisection based algorithm to solve the spherical one-step SLAM problem

The following is a new algorithm to obtain the optimal solution to the one-step SLAM problem based on Theorem 2 and a global bisection approach.

Algorithm 1:

- **Step 1**, compute parameters \( A, C, z_0, z_1 \) in (7)-(8), and \( c_0, a_0, \phi_0 \) in (12)-(14), using the given odometry and observation data;
- **Step 2**, obtain the one or two minima of (15) in the interval \([z_0 - \pi, z_0 + \pi]\) by Theorem 2, Table 2 and bisection algorithm, i.e., first find the monotone increasing intervals of \( f'_1(\phi) \) in \([z_0 - \pi, z_0 + \pi]\); then use bisection to obtain the one or two minima by satisfying conditions \( f''_1(\phi) = 0, f''_1(\phi) \geq 0 \), e.g., if there is only one minimum, use bisection to obtain the minimum in the interval \([z_0 - \pi, z_0 + \pi]\), else if there are two minima, use bisection to obtain the minima in intervals \([z_0 - \pi, z_0 + \varphi_1]\) and \([z_0 + \varphi_2, z_0 + \pi]\) respectively, where \( \varphi_1, \varphi_2 \) are defined in (22);
- **Step 3**, if there are two minima, compare the values \( f_1(\phi) \) of the two minima to get the optimal solution \( \phi^*; \) if there is only one minimum, then it is the optimal \( \phi^*; \)
- **Step 4**, get all the other variables \( X_L \) as a function of \( \phi^* \), i.e., using formula (11) to get the optimal solution to the spherical one-step SLAM problem.

Remark 7 Instead of solving \( f'_1(\phi) = 0 \) using bisection method as in Algorithm 1. It is also possible to apply golden section method (e.g. (Loxton & Lin 2011)) to find the global minimum of \( f'_1(\phi) \) in (15). Algorithm 1 presents a new way to obtain the solution of spherical one-step SLAM problem by solving a one dimensional problem through bisection and a closed-form formula (11). It should be pointed out that the joining of two local maps as shown in Fig. 4 can be seen as a one-step \( n \)-feature SLAM problem if each local map is treated as an integrated observation ((Huang, Wang & Dissanayake 2008)) and the corresponding covariance matrices of the two local maps are spherical. In Fig. 4, \( n_1 \) is the number of features that appear only in local map 1.

Remark 8 For \( m \)-step SLAM problem, it can be shown that the problem is equivalent to an optimization problem of \( m \) orientation variables \( \phi_{r_1}, \cdots, \phi_{r_m} \). However, the structure of the \( m \) dimensional function becomes much more complicated than \( f_1(\phi) \) in (10) due to the involvement of product terms such as \( \sin(\phi_{r_1}) \cos(\phi_{r_2}) \). On the other hand, a multi-step SLAM problem can be solved by a sequence of map joining problems, either using the sequential map joining strategy (Huang, Wang & Dissanayake 2008) or divide-and-concur strategy (Cadena & Neira 2010). Algorithm 1 can be used in both of the two strategies to obtain approximate solutions to the SLAM problem by approximating the local map covariance matrices with spherical ones.

Remark 9 As discussed in Remark 4, it is crucial to have spherical covariance matrix for our approach. Spherical covariances mean that the uncertainty is equal in all the directions which is realistic for some situations, e.g. when sensors used are laser range finders. Other sensors, in particular stereo vision, have much larger uncertainty in the depth direction. However, under these situations, we can still approximate the covariance as spherical ones first and quickly compute an approximate solution which can be used as a good initial value for the original SLAM problem.

7 Experiment results

We use the publicly available Victoria Park dataset to demonstrate the results in this paper.
7.1 Testing the number of minima

We first demonstrate the conclusion that the spherical one-step SLAM problem has only one minimum in most realistic scenarios. We use the equivalent map joining problem with spherical covariance matrices as an example. The Victoria Park dataset was divided into two parts to build two local maps. The covariance matrices of the two local maps are set to be identity matrices. Using the local map data to compute the values of $a, C_0$ in Theorem 2, we obtain $a = 203570, C_0 = -0.0028$. Obviously, these do not satisfy conditions (16)-(21) (also refer to Fig. 3), meaning that the map joining problem has only one minimum. To check this conclusion, Gauss-Newton algorithm was used to join the maps. In an experiment with more than 100 trials, the algorithm always converges to the optimal solution from arbitrary initial guesses to the robot poses and feature locations (Huang et al. 2012).

7.2 Approximate solution to SLAM

The Victoria Park dataset was divided into 200 parts to build 200 local maps using least squares optimization. With G2O (Kuemmerle, Grisetti, Strasdat, Konolige & Burgard 2011), the building of the 200 local maps took 0.8 seconds in total.

We apply Algorithm 1 to obtain an approximate solution to the sequential map joining problem as explained in Remark 7 and Remark 8, which is also an approximate solution to the original SLAM problem. Fig. 5 shows the map joining results of the 200 local maps using the new approach proposed in this paper and the I-SLSJF approach proposed in Huang, Wang, Dissanayake & Frese (2008), respectively. The average difference in x position is 0.2494m, the average difference in y position is 0.5281m (sum of the absolute difference divided by the number of landmarks and poses), the average difference in orientation for each pose is 0.0080rad (sum of the absolute difference divided by the number of poses).

The map joining process took 29.1 seconds for I-SLSJF and 2.9 seconds with the new approach proposed in this paper. The main reasons why the new method is faster than I-SLSJF are: (i) we solved one dimensional problems 200 times by Algorithm 1 instead of solving 200 problems with increasing dimension as in I-SLSJF; (ii) the information matrix computing step is not needed in our method since we simply assume all the covariance matrix are spherical. If we also make the spherical covariance matrices assumption in I-SLSJF and do not compute the information matrix, then the computation time of I-SLSJF is 18.5 seconds. The timing results of map joining are obtained by running MATLAB code on a computer Intel(R) Pentium(R) with 2.00GHz CPU and C++ code on a Macbook Pro with 2.80GHz CPU according to Ni & Dellaert (2010). It should be noted that TSAM 2 gives the exact solution instead of an approximate solution.

Table 1 gives a summary of the different techniques compared in the experiment.

<table>
<thead>
<tr>
<th>Techniques</th>
<th>Time</th>
<th>CPU</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>G2O + I-SLSJF</td>
<td>19.3s</td>
<td>2.00GHz</td>
<td>exact</td>
</tr>
<tr>
<td>G2O + I-SLSJFIdenti</td>
<td>15.3s</td>
<td>2.00GHz</td>
<td>approximate</td>
</tr>
<tr>
<td>G2O + Algorithm 1</td>
<td>3.7s</td>
<td>2.00GHz</td>
<td>approximate</td>
</tr>
<tr>
<td>TSAM2</td>
<td>5s</td>
<td>2.80GHz</td>
<td>exact</td>
</tr>
</tbody>
</table>

“G2O + I-SLSJF” means using G2O to build the 200 local maps and using I-SLSJF to join the 200 local maps. “G2O + I-SLSJFIdentity” means the same but assuming identity covariance matrices and not computing the information matrix in I-SLSJF. “G2O + Algorithm 1” has a similar meaning. The timing of these three algorithms include 0.8s for local map building by G2O.

Fig. 5. Comparison of map joining results using the new approach and I-SLSJF approach of the Victoria Park 200 local maps data. The circle denotes the result of the new approach, and the dots denotes the result of I-SLSJF approach.

8 Conclusions and future work

The main contribution of this paper is a mathematical proof of the special properties of the one-step SLAM problem, i.e., the high dimensional SLAM problem is equivalent to a simple one-dimension optimization problem under the condition that the associated covariance matrices are spherical. The necessary and sufficient condition for the existence of one or two minima makes it possible to guarantee that the globally optimal solution has been reached, which leads to the possibility of obtaining robust solutions to the SLAM problem even when the initial guess obtained through odometry or relative pose estimates is unreliable.

More research is needed to analyze the multi-step SLAM problem, the impact of using non-spherical covariance matrices and robust kernels in the objective function (Kuemmerle et al. 2011), and the extension of the work to 3D. It is not difficult to prove that the one-step 3D SLAM is equivalent to an optimization problem in 3 variables (the three parameters describing the robot orientation). However, it is non-trivial to derive the explicit conditions for the existence of one or more minima mainly because that the 3D orientation...
link is a lot more difficult to analyze. Work in all these directions has the potential to enhance the understanding of this important robotics problem and lead to more reliable and efficient solutions to robot navigation.

9 Appendix

9.1 Proof of Theorem 2

Let \( \phi = \phi - z_0, c = a_0 c_\phi, C_\phi = z_\phi - \phi_0 \), then minimizing \( f_1(\phi) \) in (15) is equivalent to minimizing the function

\[
\tilde{f}_1(\phi) = \frac{1}{2} p_\phi f_1(\phi) = \frac{1}{2} p_\phi c_\phi - a_0 \cos(\phi + C_\phi) + \frac{1}{2} \phi^2.
\]

Let \( h(\phi) = \tilde{f}_1(\phi) \), and \( g(\phi) = \tilde{f}_1'(\phi) \), we have

\[
h(\phi) = a \sin(\phi + C_\phi) + \phi \quad g(\phi) = a \cos(\phi + C_\phi) + 1
\]

a local minimum \( \phi \) of \( \tilde{f}_1(\phi) \) satisfies

\[
h(\phi) = 0, \quad g(\phi) \geq 0 \quad (23)
\]

and analyzing the local minima \( \phi \in [-\pi, \pi] \) of \( \tilde{f}_1(\phi) \) is equivalent to analyzing the local minima \( \phi \in [z_\phi - \pi, z_\phi + \pi] \) of \( f_1(\phi) \) in (15).

First consider the case when \( a < 1 \). Since \( \cos(\phi + C_\phi) \geq -1, g(\phi) > 0 \) for any \( \phi \), which means \( h(\phi) \) is monotonically increasing. Since we have

\[
h(-\pi) = a \sin(-\pi + C_\phi) - \pi < a - \pi < 0 \quad h(\pi) = a \sin(\pi + C_\phi) + \pi > -a + \pi > 0
\]

there is only and one only \( \phi \in [-\pi, \pi] \) satisfying (23).

Now consider the case when \( a \geq 1 \). \( g(\phi) = 0 \) has two solutions in \([-\pi, \pi]\) which are \( \phi_1 = \text{wrap}(\arccos(-\frac{1}{2}) - C_\phi) \), and \( \phi_2 = \text{wrap}(\arccos(\frac{1}{2}) - C_\phi) \).

First consider the case when \( \phi_1 \leq \phi_2 \), since \( g(\phi_1) = 0, g(\phi_2) = 0, \phi_1, \phi_2 \) divide the interval \([-\pi, \pi]\) into three intervals where \( h(\phi) \) is monotone in each of them, i.e., \([-\pi, \phi_1], [\phi_1, \phi_2], \) and \([\phi_2, \pi]\), there are at most two \( \phi \) satisfying (23), since between two monotonously increasing zero-crossings of \( h(\phi) \), there must be a region where \( h(\phi) \) decreases. So there are at most two \( \phi \) satisfying (23) and this occurs only if \( h(\phi) \) is increasing in \([-\pi, \phi_1]\), decreasing in \([\phi_1, \phi_2]\), and increasing again in \([\phi_2, \pi]\).

Since we only consider \( \phi \in [-\pi, \pi] \), the number of solutions to satisfy (23) can be analyzed by observing the four values \( h(-\pi), h(\phi_1), h(\phi_2), h(\pi) \).

<table>
<thead>
<tr>
<th>( h(\phi) )</th>
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<tbody>
<tr>
<td>( h(\phi_1) )</td>
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<tr>
<td>( h(\phi_2) )</td>
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<td>( h(\pi) )</td>
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</tbody>
</table>

Note that

\[
h(-\pi) = -a \sin C_\phi - \pi, \quad h(\pi) = -a \sin C_\phi + \pi \quad h(\phi_1) = \sqrt{a^2 - 1} + \phi_1, \quad h(\phi_2) = -\sqrt{a^2 - 1} + \phi_2
\]

it is impossible to have \( h(-\pi) > 0 \) and \( h(\pi) < 0 \) simultaneously. It can also be proved that the cases

\[
h(-\pi) < 0, h(\phi_1) < 0, h(\phi_2) < 0, h(\pi) < 0 \quad h(-\pi) > 0, h(\phi_1) > 0, h(\phi_2) > 0, h(\pi) > 0
\]

could never happen. Thus there are only 10 cases which are stated in Table 2.

Similarly, it can be proved that there is at most one solution to (23) for the case when \( \phi_1 > \phi_2 \).

Thus, there are two solutions to (23) if and only if conditions in (17)-(20) hold simultaneously together with \( \phi_1 < \phi_2 \) and \( a > 1 \). This completes the proof.

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