

L^∞ -Bounded Robust Control of Nonlinear Cascade Systems ¹

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Abstract

In this paper, we consider the L^∞ -bounded robust control problem for a class of nonlinear cascade systems with disturbances. Sufficient conditions are provided under which a hard bound is imposed on the system performance measure. The backstepping approach is used for controller design. Examples are provided to illustrate the method.

Key-words. Nonlinear cascade systems, L^∞ bound, robust control, backstepping.

1 Introduction

In the design of robust control systems, L^∞ -type (l^∞ -type) criteria are used when a hard bound on the system performance measure is required. Some recent work in this area is described in the references [1, 2, 4, 5, 15]. For example, in our re-

cent work [5], l^∞ robustness analysis and synthesis problems for general nonlinear systems were studied; in particular, necessary and sufficient conditions are provided, and a controller design procedure is given in terms of dynamic programming equations (or inequalities). However, solving dynamic programming equations for high order systems is computationally complex, and this motivates us to look for constructive controller design methods for nonlinear cascade systems with some special structure, as we discuss in this paper.

Recently, considerable attention has been paid to robust control problems for nonlinear cascade systems with strict-feedback form [10]. Some effective techniques for the construction of feedback control laws (e.g. backstepping) were developed exploiting the special structure of these systems. Different performance requirements have been considered, such as L_2 gain disturbance rejection with internal stability [12, 6, 7], input to state stability [9, 10], integral input to state stability [11], both local optimality and global inverse optimality [3], etc.

In this paper, we consider the L^∞ -bounded robust control problem for nonlinear cascade sys-

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tems with strict-feedback form. The disturbance inputs are assumed to be bounded. By assuming the L^∞ -bounded (LIB) dissipation property [5] for the low order closed loop system, we provide a controller design method for the higher order cascade system such that the closed loop system is LIB dissipative. The popular backstepping technique [10] is adapted to this L^∞ context, and is used for the construction of the feedback controller.

This paper is organized as follows. In Section 2 the L^∞ -bounded robust control problem for nonlinear cascade systems to be solved is formulated, and some preliminary results are given. In Section 3, the solution of the problem and its proof are given and the issue of asymptotic stability is considered. In Section 4 we present two simple examples to illustrate the application of the backstepping method, and some concluding remarks are provided in Section 5.

2 Problem Statement

We consider a nonlinear cascade system of the form

$$\begin{cases} \dot{x} = f_1(x) + g_1(x)w + g_2(x)y \\ \dot{y} = u + f_2(x, y)w \\ z = g(x), \end{cases} \quad (1)$$

where $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ are the states, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{W} \subset \mathbf{R}^r$ is the disturbance input, and $z \in \mathbf{R}$ is the performance quantity.

Assumption 2.1 *The set $\mathbf{W} \subset \mathbf{R}^r$ is bounded; functions f_1, g_1, g_2, f_2 are locally Lipschitz continuous; for any locally Lipschitz continuous control law $u = \bar{\alpha}(x, y)$, the closed loop system is a forward complete system, meaning that for any initial state and any disturbance input, the solution is defined on the entire interval $[0, +\infty)$.*

Throughout this paper, we denote

$$d = \sup_{w \in \mathbf{W}} |w| < +\infty \quad (2)$$

where $|\cdot|$ is the Euclidean norm. Also, for $0 \leq t_1 < t_2 \leq +\infty$, we denote by \mathcal{W}_{t_1, t_2} the class of \mathbf{W} -valued disturbance inputs defined on the time interval $[t_1, t_2]$.

Problem. Given a set $\bar{B}_0 \subset \mathbf{R}^{n+m}$, we wish to find, if possible, a state feedback controller

$$u = \bar{\alpha}(x, y)$$

such that the resulting closed loop system is L^∞ bounded (LIB) dissipative with respect to \bar{B}_0 , i.e. there exists $\bar{\beta} : \bar{B}_0 \rightarrow \mathbf{R}$ such that for the closed loop system ((1) and $u = \bar{\alpha}(x, y)$)

$$z(t) \leq \bar{\beta}(x_0, y_0), \quad \forall (x_0, y_0) \in \bar{B}_0, \forall w_{0,t} \in \mathcal{W}_{0,t}, \forall t \geq 0. \quad (3)$$

Remark 2.2 The property (3) concerns the boundedness of a function of trajectories, and cover asymptotically stable, stable and limit cycle behavior. Combining the property (3) with asymptotic stability property is a stronger requirement and will be discussed in the later part of Section 3.

We solve this problem by using the popular backstepping technique [10] to construct the required state feedback controller $\bar{\alpha}(x, y)$. To this end, we consider the following subsystem

$$\begin{cases} \dot{x} = f_1(x) + g_1(x)w + g_2(x)u \\ z = g(x). \end{cases} \quad (4)$$

In the spirit of backstepping, it is natural to assume that this subsystem enjoys the desired property, which here means the existence of a state feedback controller $\alpha(x)$, a set $B_0 \subset \mathbf{R}^n$, and a function $\beta : B_0 \rightarrow \mathbf{R}$ such that for the closed loop system ((4) and $u = \alpha(x)$),

$$z(t) \leq \beta(x_0), \quad \forall x_0 \in B_0, \forall w_{0,t} \in \mathcal{W}_{0,t}, \forall t \geq 0. \quad (5)$$

However, this is not enough, and in fact we need the following stronger assumption which, as we shall see (Lemma 2.5), implies (5) for the subsystem. The assumption is an extension of the dissipative system framework developed in [5] for LIB problems.

To specify this assumption, we need some notation. For a function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ and a number $\delta \leq +\infty$, denote

$$\begin{aligned} S_\delta^V &= \{x \in \mathbf{R}^n : V(x) < \delta\}, \\ \bar{S}_\delta^V &= \{x \in \mathbf{R}^n : V(x) \leq \delta\}. \end{aligned} \quad (6)$$

Assumption 2.3 *There exist a C^1 function $\alpha : \mathbf{R}^n \rightarrow \mathbf{R}^m$, a C^1 function $V : \mathbf{R}^n \rightarrow \mathbf{R}$, two numbers ρ, δ with $\delta < \rho \leq +\infty$, and two positive real numbers $\eta > 0, \Delta > 0$, such that*

$$\begin{aligned} V(x) &\geq g(x), \quad \forall x \in \bar{S}_\rho^V, \\ \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] &\leq \Delta, \\ &\quad \forall x \in \bar{S}_\rho^V, \forall w \in \mathbf{W}, \\ \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] &\leq -\eta, \\ &\quad \forall x \in \bar{S}_\rho^V - S_\delta^V, \forall w \in \mathbf{W} \end{aligned} \quad (7)$$

where $\nabla V(x)$ is the gradient of V and $\bar{S}_\rho^V, S_\delta^V$ are defined by (6).

Remark 2.4 Since $\delta < \rho$, the set $\bar{S}_\rho^V - S_\delta^V$ has non-empty interior. Also, the shape and size of \bar{S}_ρ^V depend on the function V and the number ρ .

Lemma 2.5 Under Assumptions 2.1 and 2.3, the closed loop system ((4) and $u = \alpha(x)$) is LIB dissipative with respect to $B_0 = \bar{S}_\rho^V$. In particular, (5) holds for $\beta : B_0 \rightarrow \mathbf{R}$ defined by

$$\beta(x_0) = \begin{cases} \delta & \text{if } x_0 \in \bar{S}_\delta^V \\ V(x_0) & \text{if } x_0 \in \bar{S}_\rho^V - \bar{S}_\delta^V \end{cases} \quad (8)$$

Proof: By condition (7) in Assumption 2.3, we have

$$\begin{aligned} V(x) &\geq g(x), \quad \forall x \in \bar{S}_\rho^V, \\ \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] &< 0, \quad (9) \\ \forall x \in \bar{S}_\rho^V - S_\delta^V, \forall w \in \mathbf{W}. \end{aligned}$$

Let $w_{0,\infty} \in \mathcal{W}_{0,\infty}$, $x_0 \in B_0 = S_\rho^V$, and denote by $x(t)$, $t \geq 0$ the resulting trajectory of the closed loop system ((4) and $u = \alpha(x)$).

We now show that if $x_0 \in \bar{S}_\delta^V$, then $x(t) \in \bar{S}_\delta^V$ for all $t \geq 0$. Suppose not, i.e. there exists $t_2 > t_1 > 0$ such that $x(t_1) \in \bar{S}_\delta^V - S_\delta^V$ and $x(t) \in \bar{S}_\rho^V - \bar{S}_\delta^V$ for all $t_1 < t < t_2$, so that

$$V(x(t)) > \delta.$$

It then follows from the second line of (9) that

$$\begin{aligned} V(x(t)) &= V(x(t_1)) + \int_{t_1}^t \nabla V(x(s))[f_1(x(s)) \\ &\quad + g_1(x(s))w(s) + g_2(x(s))\alpha(x(s))]ds \\ &< V(x(t_1)) = \delta \end{aligned}$$

for all $t_1 < t < t_2$. This is a contradiction, and so we must have $x(t) \in \bar{S}_\delta^V$ for all $t \geq 0$.

Therefore if $x_0 \in \bar{S}_\delta^V$, we have

$$z(t) = g(x(t)) \leq V(x(t)) \leq \delta = \beta(x_0)$$

for all $t \geq 0$. Here β is defined in (8).

Similarly, we can prove that if $x_0 \in \bar{S}_\rho^V$, then $x(t) \in \bar{S}_\rho^V$ for all $t \geq 0$ (make use of the second line of (9) and the fact that V is a C^1 function). Now suppose that $x_0 \in \bar{S}_\rho^V - \bar{S}_\delta^V$. Then

$$\delta < V(x_0) \leq \rho.$$

So we have either case (i), $x(t) \in \bar{S}_\rho^V - \bar{S}_\delta^V$ for all $t \geq 0$, or case (ii), there exists $t_1 > 0$ such that $x(t_1) \in \bar{S}_\delta^V - S_\delta^V$ and $x(t) \in \bar{S}_\rho^V - \bar{S}_\delta^V$ for all $0 \leq t < t_1$.

In case (i), we have

$$\begin{aligned} z(t) &= g(x(t)) \leq V(x(t)) \\ &= V(x_0) + \int_0^t \nabla V(x(s))[f_1(x(s)) \\ &\quad + g_1(x(s))w(s) + g_2(x(s))\alpha(x(s))]ds \\ &\leq V(x_0) = \beta(x_0) \end{aligned} \quad (10)$$

for all $t \geq 0$.

In case (ii), (10) holds for all $0 \leq t \leq t_1$. For $t > t_1$, we have $x(t) \in \bar{S}_\delta^V$ and hence

$$z(t) = g(x(t)) \leq V(x(t)) \leq \delta < V(x_0) = \beta(x_0).$$

This completes the proof. \blacksquare

Remark 2.6 The condition in Assumption 2.3 is somewhat similar to the Lyapunov characterization of Input-to-State Stability (ISS) property [14, 13], where the ISS Lyapunov function V satisfies

$$\dot{V}(x) \leq -\gamma_1(|x|) + \gamma_2(|w|), \quad \forall w \in \mathbf{R}^r, \forall x \in \mathbf{R}^n \quad (11)$$

for some class \mathcal{K}_∞ functions γ_1, γ_2 . In fact, since we only consider bounded disturbances with $|w| \leq d$ (see (2)), the ISS Lyapunov function V satisfies

$$\dot{V}(x) \leq -\gamma_1(|x|) + \gamma_2(d), \quad \forall w \in \mathbf{W}, \forall x \in \mathbf{R}^n. \quad (12)$$

Hence if we choose $\delta > 0$ such that $\eta = \gamma_1(\delta) - \gamma_2(d) > 0$, then we have

$$\begin{aligned} \dot{V}(x) &\leq \gamma_2(d), \quad \forall w \in \mathbf{W}, \forall x \in \mathbf{R}^n, \\ \dot{V}(x) &\leq -\eta, \quad \forall w \in \mathbf{W}, \forall |x| \geq \delta, \end{aligned} \quad (13)$$

which is similar to the condition in Assumption 2.3.

3 Solution to the Problem

The following theorem shows that the backstepping approach is successful in solving the LIB controller synthesis problem described in §2 for system (1). The following notation is used: for a function $\bar{V} : \mathbf{R}^{n+m} \rightarrow \mathbf{R}$ and a number $\delta \leq +\infty$, denote

$$\begin{aligned} S_\delta^{\bar{V}} &= \{(x, y) \in \mathbf{R}^{n+m} : \bar{V}(x, y) < \delta\}, \\ \bar{S}_\delta^{\bar{V}} &= \{(x, y) \in \mathbf{R}^{n+m} : \bar{V}(x, y) \leq \delta\}. \end{aligned} \quad (14)$$

Theorem 3.1 Given $\bar{B}_0 \subset \mathbf{R}^{n+m}$, assume Assumptions 2.1 and 2.3 hold. Then there exists a state feedback controller $u = \bar{\alpha}(x, y)$ such that the

closed loop system ($u = \bar{\alpha}(x, y)$ and (1)) is LIB dissipative with respect to \bar{B}_0 provided that

$$\bar{B}_0 \subset \bar{S}_\rho^{\bar{V}}, \quad (15)$$

where $\bar{V} : \mathbf{R}^{n+m} \rightarrow \mathbf{R}$ is defined by

$$\bar{V}(x, y) \triangleq V(x) + \frac{1}{2}[y - \alpha(x)]^T [y - \alpha(x)]. \quad (16)$$

i.e. there exists a function $\bar{\beta} : \bar{B}_0 \rightarrow \mathbf{R}$ such that (3) holds. Indeed, the items $\bar{\alpha}(x, y)$ and $\bar{\beta}$ are constructed from the functions V, \bar{V} , the subsystem feedback $\alpha(x)$ and several parameters in Assumption 2.3 as follows:

1. Fix $0 < \varepsilon < \rho - \delta$, where ρ and δ specified in Assumption 2.3, and define

$$\begin{aligned} \bar{\alpha}(x, y) & \triangleq -g_2^T(x) \nabla V^T(x) + \nabla \alpha(x) [f_1(x) + g_2(x)y] \\ & \quad - [y - \alpha(x)] \left(\frac{\Delta + \eta}{2\varepsilon} + \frac{d^2}{\eta} |\nabla \alpha(x) g_1(x)|^2 \right) \\ & \quad + \frac{d^2}{\eta} |f_2(x, y)|^2, \end{aligned} \quad (17)$$

where also Δ and η are specified in Assumption 2.3.

2. Define $\bar{\beta} : \bar{B}_0 \rightarrow \mathbf{R}$ by

$$\bar{\beta}(x, y) \triangleq \begin{cases} \delta + \varepsilon, & \text{if } (x, y) \in S_{\delta+\varepsilon}^{\bar{V}}, \\ \bar{V}(x, y), & \text{if } (x, y) \in \bar{S}_\rho^{\bar{V}} - S_{\delta+\varepsilon}^{\bar{V}}. \end{cases} \quad (18)$$

Remark 3.2 Notice that the maximal sets on which the closed loop system ((4) and $u = \alpha(x)$) and the closed loop system ($u = \bar{\alpha}(x, y)$ and (1)) are LIB dissipative are $B_0 = \bar{S}_\rho^{\bar{V}}$ and $\bar{B}_0 = \bar{S}_\rho^{\bar{V}}$, respectively. By (16), the projection of $\bar{S}_\rho^{\bar{V}}$ on the x subspace is $\bar{S}_\rho^{\bar{V}}$.

In order to prove the above theorem, we use the following lemma.

Lemma 3.3 Under Assumptions 2.1 and 2.3, for any $0 < \varepsilon < \rho - \delta$, there exist a function $\bar{\alpha}(x, y)$ such that the function $\bar{V}(x, y)$ defined by (16) satisfies

$$\bar{V}(x, y) \geq g(x), \quad \forall (x, y) \in \bar{S}_\rho^{\bar{V}},$$

$$\begin{aligned} \nabla_x \bar{V}(x, y) [f_1(x) + g_1(x)w + g_2(x)y] \\ + \nabla_y \bar{V}(x, y) [\bar{\alpha}(x, y) + f_2(x, y)w] \leq \Delta + \frac{\eta}{2}, \\ \forall (x, y) \in \bar{S}_\rho^{\bar{V}}, \forall w \in \mathbf{W}, \end{aligned}$$

$$\begin{aligned} \nabla_x \bar{V}(x, y) [f_1(x) + g_1(x)w + g_2(x)y] \\ + \nabla_y \bar{V}(x, y) [\bar{\alpha}(x, y) + f_2(x, y)w] \leq -\frac{\eta}{2}, \\ \forall (x, y) \in \bar{S}_\rho^{\bar{V}} - S_{\delta+\varepsilon}^{\bar{V}}, \forall w \in \mathbf{W}. \end{aligned} \quad (19)$$

Proof: By (16), we have

$$\bar{V}(x, y) \geq V(x) \geq g(x), \quad \forall x \in \bar{S}_\rho^{\bar{V}}, \forall y \in \mathbf{R}^m, \quad (20)$$

proving the first line of (19).

Next, we evaluate the derivative of $\bar{V}(x, y)$ along the trajectory of system (1) with control u as follows:

$$\begin{aligned} \dot{\bar{V}}(x, y) & = \nabla V(x) \dot{x} + [y - \alpha(x)]^T [\dot{y} - \nabla \alpha(x) \dot{x}] \\ & = \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)y] \\ & \quad + [y - \alpha(x)]^T \{u + f_2(x, y)w \\ & \quad - \nabla \alpha(x) [f_1(x) + g_1(x)w + g_2(x)y]\} \\ & = \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad + \nabla V(x) g_2(x) [y - \alpha(x)] \\ & \quad + [y - \alpha(x)]^T \{u + f_2(x, y)w \\ & \quad - \nabla \alpha(x) [f_1(x) + g_1(x)w + g_2(x)y]\} \\ & = \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad + [y - \alpha(x)]^T g_2^T(x) \nabla V^T(x) \\ & \quad + [y - \alpha(x)]^T \{u + f_2(x, y)w \\ & \quad - \nabla \alpha(x) [f_1(x) + g_1(x)w + g_2(x)y]\}. \end{aligned} \quad (21)$$

Now choose

$$\begin{aligned} u & = \bar{\alpha}(x, y) \\ & \triangleq -g_2^T(x) \nabla V^T(x) + \nabla \alpha(x) [f_1(x) + g_2(x)y] \\ & \quad - [y - \alpha(x)] (c_1 + c_2 |\nabla \alpha(x) g_1(x)|^2 \\ & \quad + c_2 |f_2(x, y)|^2), \end{aligned} \quad (22)$$

where c_1, c_2 will be decided shortly. Then we have

$$\begin{aligned} \dot{\bar{V}}(x, y) & = \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad + [y - \alpha(x)]^T \{-[y - \alpha(x)] (c_1 \\ & \quad + c_2 |\nabla \alpha(x) g_1(x)|^2 + c_2 |f_2(x, y)|^2) \\ & \quad - \nabla \alpha(x) g_1(x)w + f_2(x, y)w\} \\ & = \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad - c_1 |y - \alpha(x)|^2 \\ & \quad - \{c_2 |y - \alpha(x)|^2 |\nabla \alpha(x) g_1(x)|^2 \\ & \quad + [y - \alpha(x)]^T \nabla \alpha(x) g_1(x)w\} \\ & \quad - \{c_2 |y - \alpha(x)|^2 |f_2(x, y)|^2 \\ & \quad - [y - \alpha(x)]^T f_2(x, y)w\} \\ & \leq \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad - c_1 |y - \alpha(x)|^2 + \frac{1}{4c_2} |w|^2 + \frac{1}{4c_2} |w|^2 \\ & = \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad - c_1 |y - \alpha(x)|^2 + \frac{1}{2c_2} |w|^2 \\ & \leq \nabla V(x) [f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ & \quad - c_1 |y - \alpha(x)|^2 + \frac{1}{2c_2} d^2 \end{aligned} \quad (23)$$

Here, we have used the bound $|w| \leq d$ in the last step (see (2)).

Now fix $0 < \varepsilon < \rho - \delta$, where ρ and δ specified in Assumption 2.3, and let $(x, y) \in \bar{S}_\rho^{\bar{V}} - S_{\delta+\varepsilon}^{\bar{V}}$. Then

$$\rho \geq \bar{V}(x, y) = V(x) + \frac{1}{2}[y - \alpha(x)]^T [y - \alpha(x)] \geq \delta + \varepsilon,$$

and hence either case (i) $\rho \geq V(x) \geq \delta$, or case (ii) $\rho \geq \frac{1}{2}[y - \alpha(x)]^T[y - \alpha(x)] \geq \varepsilon$.

Case (i) If $\rho \geq V(x) \geq \delta$, then $x \in \bar{S}_\rho^V - S_\delta^V$, and hence by Assumption 2.3 we have

$$\nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \leq -\eta,$$

so we have

$$\dot{\bar{V}}(x, y) \leq -\eta - c_1|y - \alpha(x)|^2 + \frac{1}{2c_2}d^2.$$

If we choose

$$c_2 = \frac{d^2}{\eta}, \quad (24)$$

then we have

$$\dot{\bar{V}}(x, y) \leq -\frac{\eta}{2} - c_1|y - \alpha(x)|^2 \leq -\frac{\eta}{2}.$$

Hence we choose c_2 as (24).

Case (ii) Now since $V(x) \leq \bar{V}(x, y) \leq \rho$, we have $x \in \bar{S}_\rho^V$, and so by Assumption 2.3,

$$\nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \leq \Delta,$$

and hence

$$\begin{aligned} \dot{\bar{V}}(x, y) &\leq \Delta - c_1|y - \alpha(x)|^2 + \frac{1}{2c_2}d^2 \\ &= \Delta - c_1|y - \alpha(x)|^2 + \frac{\eta}{2}. \end{aligned}$$

If $\rho \geq \frac{1}{2}[y - \alpha(x)]^T[y - \alpha(x)] = \frac{1}{2}|y - \alpha(x)|^2 \geq \varepsilon$, and if we choose

$$c_1 = \frac{\Delta + \eta}{2\varepsilon}, \quad (25)$$

then we have

$$\dot{\bar{V}}(x, y) \leq \Delta - 2c_1\varepsilon + \frac{\eta}{2} \leq -\frac{\eta}{2}.$$

Hence we choose c_1 as (25).

Therefore with c_1 and c_2 as (25) and (24) we have shown that

$$\dot{\bar{V}}(x, y) \leq -\frac{\eta}{2} \quad (26)$$

for $(x, y) \in \bar{S}_\rho^V - S_{\delta+\varepsilon}^V$. This proves the last line of (19).

Finally, suppose $(x, y) \in \bar{S}_\rho^V$, so that $\bar{V}(x, y) \leq \rho$. Then we have $V(x) \leq \rho$, by (23) and hence

$$\dot{\bar{V}}(x, y) \leq \Delta + \frac{\eta}{2}.$$

Thus $\bar{V}(x, y)$ satisfies the second line of (19), and the proof is complete. ■

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: The function $\bar{\alpha}(x, y)$ in (17) is determined by (22) and (25), (24). By Lemma 3.3, if we denote $\bar{\rho} = \rho$, $\bar{\delta} = \delta + \varepsilon$, $\bar{\eta} = \frac{\eta}{2} > 0$, $\bar{\Delta} = \Delta + \frac{\eta}{2} > 0$, then the function $\bar{V}(x, y)$ satisfies

$$\begin{aligned} \bar{V}(x, y) &\geq g(x), \quad \forall (x, y) \in \bar{S}_\rho^V, \\ \nabla_x \bar{V}(x, y)[f_1(x) + g_1(x)w + g_2(x)y] \\ &+ \nabla_y \bar{V}(x, y)[\bar{\alpha}(x, y) + f_2(x, y)w] \leq \bar{\Delta}, \\ \forall (x, y) &\in \bar{S}_\rho^V, \forall w \in \mathbf{W}, \end{aligned} \quad (27)$$

$$\begin{aligned} \nabla_x \bar{V}(x, y)[f_1(x) + g_1(x)w + g_2(x)y] \\ &+ \nabla_y \bar{V}(x, y)[\bar{\alpha}(x, y) + f_2(x, y)w] \leq -\bar{\eta}, \\ \forall (x, y) &\in \bar{S}_\rho^V - S_\delta^V, \forall w \in \mathbf{W}. \end{aligned}$$

The proof now follows using similar arguments to the proof of Lemma 2.5.

Remark 3.4 By Lemma 3.3 and the proof of Theorem 3.1, $\bar{V}(x, y)$ has a similar property as that in Assumption 2.3, so we can design recursively the controller achieving LIB dissipation for higher dimensional nonlinear cascade systems with strict-feedback form [10] using similar arguments to Theorem 3.1.

We now show that under some additional assumptions, we can obtain the asymptotic stability of the closed loop when $w = 0$.

Assumption 3.5 In system (1), $f_1(0) = 0, g(0) = 0$; the C^1 function $\alpha : \mathbf{R}^n \rightarrow \mathbf{R}^m$ in Assumption 2.3 satisfies $\alpha(0) = 0$; the C^1 function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ in Assumption 2.3 satisfies

$$\begin{aligned} V(0) &= 0, \quad V(x) > 0, \quad \forall x \in \bar{S}_\rho^V, x \neq 0, \\ \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] &< 0, \quad \forall x \in \bar{S}_\rho^V, x \neq 0, \end{aligned} \quad (28)$$

where ρ is given in Assumption 2.3.

Under Assumptions 2.1, 2.3 and 3.5, Lemma 2.5 can be strengthened as follows.

Lemma 3.6 Under Assumptions 2.1, 2.3 and 3.5, the closed loop system ((4) and $u = \alpha(x)$) is LIB dissipative with respect to $B_0 = \bar{S}_\rho^V$. Moreover, when $w = 0$, the closed loop system is asymptotically stable provided that $x_0 \in \bar{S}_\rho^V$.

Proof: The LIB dissipation property is given in Lemma 2.5. The asymptotic stability property can be obtained by a standard Lyapunov stability theorem. (e.g. Theorem 3.1 in [8]). ■

By Theorem 3.1, we have the following corollary.

Corollary 3.7 Given $\bar{B}_0 \subset \mathbf{R}^{n+m}$, assume Assumptions 2.1, 2.3 and 3.5 hold. Then there exists a state feedback controller $u = \bar{\alpha}(x, y)$ such that the closed loop system ($u = \bar{\alpha}(x, y)$ and (1)) is LIB dissipative with respect to \bar{B}_0 provided that

$$\bar{B}_0 \subset \bar{S}_\rho^{\bar{V}}, \quad (29)$$

where $\bar{V} : \mathbf{R}^{n+m} \rightarrow \mathbf{R}$ is defined by (16). Furthermore, when $w = 0$, the closed loop system is asymptotically stable provided $(x_0, y_0) \in \bar{S}_\rho^{\bar{V}}$.

Proof: The LIB dissipation property is proved in Theorem 3.1. Now we show the asymptotic stability when $w = 0$. By Lyapunov stability theorem (e.g. Theorem 3.1 in [8]), we only need to prove that the $\bar{V}(x, y)$ defined by (16) satisfies

$$\begin{aligned} V(0, 0) &= 0, \\ V(x, y) &> 0, \quad \forall (x, y) \in \bar{S}_\rho^{\bar{V}}, (x, y) \neq (0, 0), \\ \dot{\bar{V}}(x, y) &= \nabla_x \bar{V}(x, y)[f_1(x) + g_2(x)y] \\ &\quad + \nabla_y \bar{V}(x, y)\bar{\alpha}(x, y) < 0, \\ \forall (x, y) &\in \bar{S}_\rho^{\bar{V}}, (x, y) \neq (0, 0). \end{aligned} \quad (30)$$

The first line of (30) is obvious since $\alpha(0) = 0$. When $w = 0$, the derivative of $\bar{V}(x, y)$ along the trajectory of system (1) with control u is:

$$\begin{aligned} \dot{\bar{V}}(x, y) &= \nabla V(x)\dot{x} + [y - \alpha(x)]^T [\dot{y} - \nabla\alpha(x)\dot{x}] \\ &= \nabla V(x)[f_1(x) + g_2(x)y] \\ &\quad + [y - \alpha(x)]^T \{u - \nabla\alpha(x)[f_1(x) + g_2(x)y]\} \\ &= \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] \\ &\quad + \nabla V(x)g_2(x)[y - \alpha(x)] \\ &\quad + [y - \alpha(x)]^T \{u - \nabla\alpha(x)[f_1(x) + g_2(x)y]\} \\ &= \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] \\ &\quad + [y - \alpha(x)]^T g_2^T(x)\nabla V^T(x) \\ &\quad + [y - \alpha(x)]^T \{u - \nabla\alpha(x)[f_1(x) + g_2(x)y]\}. \end{aligned} \quad (31)$$

With the controller (22), we have

$$\begin{aligned} \dot{\bar{V}}(x, y) &= \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] \\ &\quad + [y - \alpha(x)]^T \{-[y - \alpha(x)](c_1 \\ &\quad + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x, y)|^2)\} \\ &= \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] - (c_1 \\ &\quad + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x, y)|^2)|y - \alpha(x)|^2 \end{aligned} \quad (32)$$

Suppose $(x, y) \in \bar{S}_\rho^{\bar{V}}$, then $x \in \bar{S}_\rho^{\bar{V}}$. If $x \neq 0$, then by (28),

$$\dot{\bar{V}}(x, y) \leq \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] < 0.$$

If $x = 0$, $y \neq \alpha(0) = 0$, then

$$\dot{\bar{V}}(x, y) \leq -(c_1 + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x, y)|^2)|y - \alpha(x)|^2 < 0.$$

Hence the second line of (30) holds and the proof is completed. \blacksquare

4 Illustrative Examples

Example 1. Consider two-dimensional system

$$\begin{cases} \dot{x} = x^2 + w + y \\ \dot{y} = u + w \\ z = |x| \end{cases} \quad (33)$$

where $x, y, z, u \in \mathbf{R}$ and

$$w \in \mathbf{W} = \{w \in \mathbf{R} : |w| \leq 1\}.$$

The subsystem is

$$\begin{cases} \dot{x} = x^2 + w + u \\ z = |x| \end{cases} \quad (34)$$

Choose controller

$$u = \alpha(x) = -x - x^2, \quad (35)$$

then the closed-loop subsystem is

$$\begin{cases} \dot{x} = -x + w \\ z = |x| \end{cases} \quad (36)$$

The system (36) is LIB dissipative with respect to $B_0 = \mathbf{R}$ because we can prove that

$$z(t) = |x(t)| \leq \beta_a(x_0) \triangleq \max\{|x_0|, 1\}, \quad \forall x_0 \in \mathbf{R}, \forall t \geq 0, \forall w_{0,t} \in \mathcal{W}_{0,t}. \quad (37)$$

In fact, the solution of this system is

$$x(t) = e^{-t}x_0 + \int_0^t e^{s-t}w(s)ds. \quad (38)$$

Hence

$$\begin{aligned} |x(t)| &\leq e^{-t}|x_0| + e^{-t} \int_0^t |e^s w(s)| ds \\ &\leq e^{-t}|x_0| + e^{-t} \int_0^t e^s ds \\ &= e^{-t}|x_0| + e^{-t}(e^t - 1) \\ &= e^{-t}(|x_0| - 1) + 1 \end{aligned}$$

and

$$\sup_{t \geq 0} |x(t)| = \max\{|x_0|, 1\}. \quad (39)$$

We choose any C^1 function $V : \mathbf{R} \rightarrow \mathbf{R}$ which satisfies

$$\begin{aligned} V(x) &= |x|, \quad \forall x \in \mathbf{R} - [-0.5, 0.5]; \\ V(x) &\geq |x|, \quad \dot{V}(x) \leq 1, \quad \forall x \in \mathbf{R}. \end{aligned} \quad (40)$$

For any $\eta > 0$, choose

$$\delta = 1 + \eta,$$

choose $\rho = +\infty$ and $\Delta = 2$, then $V(x)$ satisfies the condition in Assumption 2.3. By Lemma 2.5, system (36) is LIB dissipative with respect to $B_0 = \bar{S}_\rho^V = \mathbf{R}$. The function β defined by (8) is

$$\beta(x) = \max\{|x|, 1 + \eta\}, \quad \forall x \in \mathbf{R},$$

which is larger than the minimal function β_a defined in (37) (but can be made as close as we want by choosing a small η).

Now for any $\varepsilon > 0$, choose the controller for system (33) as

$$\begin{aligned} \bar{\alpha}(x, y) &= -g_2^T(x) \nabla V^T(x) + \nabla \alpha(x) [f_1(x) + g_2(x)y] \\ &\quad - [y - \alpha(x)] (c_1 + c_2 |\nabla \alpha(x) g_1(x)|^2 \\ &\quad + c_2 |f_2(x, y)|^2) \\ &= -V'(x) - (1 + 2x)(x^2 + y) - (y + x + x^2) \\ &\quad \cdot (c_1 + c_2(1 + 2x)^2 + c_2) \\ &= -V'(x) - (1 + 2x)(x^2 + y) - (y + x + x^2) \\ &\quad \cdot \left(\frac{\Delta + \eta}{2\varepsilon} + \frac{d^2}{\eta} (1 + 2x)^2 + \frac{d^2}{\eta} \right) \\ &= -V'(x) - (1 + 2x)(x^2 + y) - (y + x + x^2) \\ &\quad \cdot \left(\frac{2 + \eta}{2\varepsilon} + \frac{1}{\eta} (1 + 2x)^2 + \frac{1}{\eta} \right). \end{aligned}$$

then the closed-loop system is L^∞ -bounded dissipative on $\bar{B}_0 = \mathbf{R}^2$. Moreover, since $\bar{V}(x, y)$ is defined by

$$\begin{aligned} \bar{V}(x, y) &= V(x) + \frac{1}{2} [y - \alpha(x)]^T [y - \alpha(x)] \\ &= V(x) + \frac{1}{2} (y + x + x^2)^2, \quad \forall (x, y) \in \mathbf{R}^2, \end{aligned}$$

the solution of the closed-loop system

$$\begin{cases} \dot{x} = x^2 + w + y \\ \dot{y} = \bar{\alpha}(x, y) + w \\ z = |x| \end{cases} \quad (41)$$

satisfies

$$\begin{aligned} z(t) &= |x(t)| \leq \bar{\beta}(x_0, y_0), \\ \forall t \geq 0, \forall w_{0,t} \in \mathcal{W}_{0,t}, \forall (x_0, y_0) \in \mathbf{R}^2, \end{aligned} \quad (42)$$

where $\bar{\beta}(x, y)$ is defined by

$$\bar{\beta}(x, y) = \begin{cases} 1 + \eta + \varepsilon, & \forall (x, y) \in S_{1+\eta+\varepsilon}^{\bar{V}}, \\ V(x) + \frac{1}{2} (y + x + x^2)^2, & \forall (x, y) \notin S_{1+\eta+\varepsilon}^{\bar{V}}. \end{cases}$$

Example 2. Consider system

$$\begin{cases} \dot{x} = x^2 + w + y \\ \dot{y} = u + w \\ z = x^2 \end{cases} \quad (43)$$

The only difference between systems (33) and (43) is the performance z .

With controller (35), the closed-loop subsystem is

$$\begin{cases} \dot{x} = -x + w \\ z = x^2 \end{cases} \quad (44)$$

The system (44) is LIB dissipative with respect to $B_0 = \mathbf{R}$ because

$$\begin{aligned} z(t) = x^2(t) &\leq \beta_a(x_0) \stackrel{\Delta}{=} \max\{x_0^2, 1\}, \\ \forall x_0 \in \mathbf{R}, \forall t \geq 0, \forall w_{0,t} \in \mathcal{W}_{0,t}. \end{aligned} \quad (45)$$

Now we choose C^1 function $V : \mathbf{R} \rightarrow \mathbf{R}$ as follows

$$V(x) = x^2, \quad \forall x \in \mathbf{R}. \quad (46)$$

For any $\eta > 0$, choose

$$\delta = \left(1 + \frac{\eta}{2}\right)^2,$$

also choose $\rho = +\infty$ and $\Delta = 2$, then $V(x)$ satisfies the condition in Assumptions 2.3 and 3.5. By Lemma 3.6, system (44) is LIB dissipative with respect to $B_0 = \bar{S}_\rho^V = \mathbf{R}$ and is asymptotically stable when $w = 0$. The function β defined by (8) is

$$\beta(x) = \max\{x^2, \delta\}, \quad \forall x \in \mathbf{R}.$$

Now for any $\varepsilon > 0$, choose

$$\begin{aligned} \bar{\alpha}(x, y) &= -V'(x) - (1 + 2x)(x^2 + y) - (y + x + x^2) \\ &\quad \cdot (c_1 + c_2(1 + 2x)^2 + c_2) \\ &= -2x - (1 + 2x)(x^2 + y) - (y + x + x^2) \\ &\quad \cdot \left(\frac{2 + \eta}{2\varepsilon} + \frac{1}{\eta} (1 + 2x)^2 + \frac{1}{\eta} \right). \end{aligned}$$

then the closed-loop system ((43) with $u = \bar{\alpha}(x, y)$) is L^∞ -bounded dissipative on $\bar{B}_0 = \mathbf{R}^2$ and is asymptotically stable when $w = 0$. It is of interest to note that this controller also achieves the L^∞ -bounded dissipativeness for the system in Example 1. Indeed, $|x(t)| \leq \sqrt{\bar{\beta}(x_0, y_0)}$ when $z(t) = x^2(t) \leq \bar{\beta}(x_0, y_0)$.

5 Conclusion

In this paper we have demonstrated the feasibility of applying the backstepping method to the design of feedback controllers in the context of L^∞ performance criteria. For systems with the special cascade and strict-feedback form, the need to solve numerically high order dynamic programming equation is avoided. Future research will consider applications of these results, as well as the development of methods for the output feedback case (c.f. [7]).

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