In Chapter 9 we found that RLC circuits possess the distinctive ability to provide natural responses of the oscillatory type. This ability stems from the flow of energy back and forth between the capacitance and the inductance. It is not surprising that RLC circuits also exhibit a distinctive ac behavior. This stems from the fact that inductance and capacitance have dual ac characteristics: The inductive reactance $X_L = \omega L$ is positive and is directly proportional to $\omega$, whereas the capacitive reactance $X_C = -1/\omega C$ is negative and inversely proportional to $\omega$. Depending on frequency, $X_L$ or $X_C$ may dominate, or the two reactances may cancel each other out, making a series LC combination act as a short circuit, and a parallel LC combination as an open. This cancellation provides the basis for a form of resonance known as unity power-factor resonance to reflect the fact that if the series or parallel LC combination is embedded in a resistive circuit, at resonance all voltages and currents in the circuit will be in phase with the applied source, thus yielding a power factor of unity.

Another distinguishing feature of resonant RLC circuits is the ability to provide a form of signal magnification known as resonance signal rise, a phenomenon not found in other passive ac circuits, where all signals in the circuit are of lesser magnitude than the applied signal.

In this chapter we first investigate the series and parallel RLC circuits, and we characterize their behavior at resonance in terms of the resonance frequency $\omega_0$ and the quality factor $Q$. We also investigate the effect of practical coils upon the $Q$ of a circuit.

Next, we examine a resonant circuit consisting of resistances and capacitances, but no inductances, and find that it always has $Q < 0.5$. However, with the inclusion of an op amp to provide a controlled amount of positive feedback, it is possible to change the $Q$ of the circuit to any value, including $Q = \infty$ or even $Q < 0$. With $Q = \infty$ the circuit yields a sustained oscillation, and with $Q < 0$ it yields a diverging oscillation.

We conclude by discussing magnitude and frequency scaling.

## 13.1 SERIES RESONANCE

In this section we investigate the phenomenon of ac resonance for the case of a capacitance and inductance connected in series. Once we reduce the remainder of the circuit to its...
AC Resonance

Thévenin equivalent, we end up with the series RLC configuration of Figure 13.1(a). The impedance seen by the source is

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + jX$$  \hspace{1cm} (13.1)

where

$$X = \omega L - \frac{1}{\omega C}$$  \hspace{1cm} (13.2)

is the net reactance of the circuit. Depending on the frequency $\omega$ of the applied source, we have three possibilities:

1. $\omega L < 1/\omega C$, or $X < 0$, indicating capacitive behavior;
2. $\omega L > 1/\omega C$, or $X > 0$, indicating inductive behavior; or
3. $\omega L = 1/\omega C$, or $X = 0$. This is the borderline between the two preceding cases. Since we now have $Z = R + j0 = R$, the RLC circuit behaves resistively, with $I$ and $V$ in phase with each other. This behavior, referred to as unity power-factor resonance, occurs at the special frequency $\omega_o$ that makes $\omega_o L = 1/\omega_o C$, or

$$\omega_o = \frac{1}{\sqrt{LC}}$$  \hspace{1cm} (13.3)

![Figure 13.1](image)

**Figure 13.1** Series RLC circuit and frequency response of its current.

For obvious reasons, $\omega_o$ is called the resonance frequency. Even though the individual impedances $Z_L$ and $Z_C$ are not zero, at resonance they cancel each other out, making the LC series combination act as a short circuit. Figure 13.2 depicts the signals in the circuit at resonance.
**Frequency Response**

We find it instructive to reexamine the frequency response of the series *RLC* circuit using phasor analysis. To simplify the formalism, we again use the quality factor of the series *RLC* circuit,

\[ Q = \frac{1}{R \sqrt{C/L}} \quad (13.4) \]

![Series RLC Circuit at Resonance](image)

If this is combined with Equation (13.3), it is straightforward to verify that the reactances can be expressed as

\[ \omega L = RQ \frac{\omega_0}{\omega} \quad (13.5 \ a) \]
\[ \frac{1}{\omega C} = RQ \frac{\omega_0}{\omega} \quad (13.5 \ b) \]

Substituting into Equation (13.1) yields the more insightful form

\[ Z = R \left[ 1 + jQ \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right] \quad (13.6) \]

The current response is readily found as \( I = V/Z \), or

\[ I = \frac{V}{R \left[ 1 + jQ \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right]} \quad (13.7) \]

Applying Phasor Rule 6 of Section 11.1 gives the amplitude and phase relationships between current and voltage as
As we know, the plot of $I_m$ versus $\omega$ for a given $V_m$ has the familiar bell-shaped profile of the **band-pass function**. Current approaches zero at both ends of the frequency spectrum because of the *open-circuit* action by $L$ at low frequencies and by $C$ at high frequencies. At the borderline frequency $\omega = \omega_o$, the reactances cancel each other out, minimizing $|Z|$ and maximizing $I_m$, so $I_m(\text{max}) = V_m/R$. At resonance the *average power* $P$ dissipated in the resistance is also maximized, and

$$P_{\text{max}} = \frac{1}{2} \frac{V_m^2}{R} = \frac{V_{\text{rms}}^2}{R}$$  (13.9)

The frequencies at which $P$ is down to half its maximum value are the *half-power frequencies* $\omega_L$ and $\omega_H$. Recall from Section 10.4 that the *half-power bandwidth* is $\text{BW} = \omega_L - \omega_H$, and that

$$\text{BW} = \frac{\omega_o}{Q}$$  (13.10)

If Equations (13.3) and (13.4) are combined, it is readily seen that

$$\text{BW} = \frac{1}{L/R}$$  (13.11)

Equations (13.3) and (13.11) indicate that in a series RLC circuit $\omega_o$ is set by $L$ and $C$, and $\text{BW}$ is set by $R$ and $L$.

**Exercise 13.1** A series RLC circuit has $R = 50 \ \Omega$, $\omega_o = 10^5$ rad/s, and $\text{BW} = 10^3$ rad/s. Find $L$ and $C$.

**ANSWER** 50 mH, 2 nF.

**Example 13.1**

A load consisting of a 20 $\Omega$ resistance in series with a 0.25 H inductance is to be driven by the utility power.
(a) Specify a suitable capacitance that, when interposed between the source and the load, will maximize power transfer.

(b) Compare with the case in which the load is connected to the utility power directly, without the coupling capacitance.

Solution

(a) We want the circuit to resonate at \( \omega_0 = 2\pi f_0 = 377 \text{ rad/s} \). Use a series capacitance \( C = \frac{1}{\omega_0^2L} = \frac{1}{(377^2 \times 0.25)} = 28.14 \mu\text{F} \). Then,

\[
P = P_{\text{max}} = \frac{V_{\text{rms}}^2}{R} = \frac{120^2}{20} = 720 \text{ W}.
\]

(b) Without the coupling capacitance, the source sees \( Z_L = R + j\omega L = 20 + j(377 \times 0.25) = 20 + j94.25 \Omega \), so \( I_{\text{rms}} = \frac{V_{\text{rms}}}{|Z_L|} = \frac{120}{\sqrt{(20^2 + 94.25^2)}} = 1.246 \text{ A} \). Hence,

\[
P = RI_{\text{rms}}^2 = 20 \times 1.246^2 = 31.03 \text{ W}.
\]

Clearly, we now have \( P \ll P_{\text{max}} \).

Phasor Diagrams

We can gain additional insight into RLC behavior by constructing the phasor diagrams for \( \omega < \omega_0 \), \( \omega = \omega_0 \), and \( \omega > \omega_0 \). It is convenient to use the current phasor \( I = I_m/\theta_I \) as the reference phasor because current is common to all three elements. The phasors of the individual element voltages are

\[
V_R = RI_m/\theta_I, \quad V_L = \omega LI_m/\theta_I + 90^\circ, \quad V_C = (1/j\omega C)I_m/\theta_I - 90^\circ.
\]

The phasor diagrams are shown in Figure 13.3. Below resonance (\( \omega < \omega_0 \)) we have \(|V_C| > |V_L|\) and \( \theta_1 > 0 \), confirming capacitive behavior. At resonance (\( \omega = \omega_0 \)) we have \( V_C = -V_L \), so \( V_R = V \) and \( \theta_1 = 0 \). The resonant circuit behaves as if it consisted of just \( R \). Above resonance (\( \omega > \omega_0 \)) we have \(|V_L| > |V_C|\) and \( \theta_1 < 0 \), confirming inductive behavior.

Resonance Voltage Rise

The phasor diagrams of Figure 13.3 and the results of Example 13.2 reveal an intriguing phenomenon, that is, depending on the operating frequency \( \omega \), the peak voltage amplitudes
across one, the other, or both of the reactive elements may exceed the peak amplitude of the applied signal. We are particularly interested in the situation at resonance, where

\[ I = \frac{V}{R/\omega_0} \]  

(13.13)

Letting \( \omega = \omega_0 \) in Equations (13.12) and using the fact that Equation (13.5) predicts \( \omega_0 L = QR \) and \( 1/\omega_0 C = QR \), the phasors of the individual element voltages are, at resonance,

\[ V_R = V_m/\theta \]  

(13.14a)

\[ V_L = QV_m/90^{\circ} \]  

(13.14b)

\[ V_C = QV_m/-90^{\circ} \]  

(13.14c)

Clearly, if \( Q > 1 \), the peak voltages across the reactive elements will be greater than that of the applied source, a phenomenon known as resonance voltage rise. The amount of magnification coincides with the quality factor of the circuit. For instance, in the preceding example we have \( Q = 2.5 \), thus confirming the values \( |V_L| = |V_C| = 2.5 |V| = 2.5 \times 10 = 25 \text{ V} \).

It is interesting to note that the smaller the resistance \( R \) in a series RLC circuit, the higher the value of \( Q \) and, hence, the greater the amount of resonance voltage rise. In the limit \( R \to 0 \) we would have, by Equation (13.4), \( Q \to \infty \), and this would result in an infinite voltage rise.

Resonance phenomena arise frequently in everyday life. A common example is offered by a child bouncing on a bed. The child exploits the ability to convert the potential energy of the bed springs into kinetic energy of his or her body, and vice versa, to achieve amazingly high bounces with little effort. The bouncing frequency at which effort is minimum is found empirically by the child: it is the resonance frequency of the mechanical system made up of the child’s body and the bed springs. Any attempt to bounce at a higher or a lower frequency than the resonance frequency would require much more effort and would result in less spectacular bounces!

Exercise 13.2 Find the peak voltage amplitudes across the coupling capacitor and the load of Example 13.1.

ANSWER 799.7 V, 817.5 V.

Exercise 13.3

(a) Using a 100 \( \mu \text{H} \) inductance, specify suitable values for \( R \) and \( C \) to implement a series RLC circuit that resonates at 1 MHz with a quality factor of 100.

(b) If the circuit is driven with a source having a peak amplitude of 1 V, find the power it dissipates at resonance, as well as the amplitudes of the voltages across its reactive elements.

ANSWER (a) 6.283 \( \Omega \), 253.3 pF; (b) 79.58 mW, 100 V.

Energy at Resonance

The instantaneous energy stored in the inductance is \( W_L(t) = (1/2)Li_L^2(t) \), and that stored in the capacitance is \( W_C(t) = (1/2)CV_C^2(t) \). The sum \( W_L(t) + W_C(t) \) is called the total stored
energy. This energy is maximum at resonance. We wish to find its value, aptly called the maximum stored energy.

At resonance, Equation (13.13) predicts \( i(t) = \frac{V_m}{R} \cos \omega_0 t \). Moreover, Equation (13.5a) allows us to write \( L = \frac{QR}{\omega_0} \) so

\[
W_L(t) = \frac{Q}{\omega_0} \frac{V_{rms}^2}{R} \cos^2 \omega_0 t \tag{13.15a}
\]

Still at resonance we have, by Equation (13.14c), \( v_c(t) = QV_m \cos(\omega_0 t - 90^\circ) = QV_m \sin \omega_0 t \). Moreover, Equation (13.5b) allows us to write \( C = \frac{1}{\omega_0 QR} \), so

\[
W_C(t) = \frac{Q}{\omega_0} \frac{V_{rms}^2}{R} \sin^2 \omega_0 t \tag{13.15b}
\]

The maximum stored energy is \( W_L(t) + W_C(t) = (Q/\omega_0)(V_{rms}^2/R)(\cos^2 \omega_0 t + \sin^2 \omega_0 t) \), or

\[
W_L(t) + W_C(t) = \frac{Q}{\omega_0} \frac{V_{rms}^2}{R} \tag{13.16}
\]

Clearly, this energy is time invariant. Physically, at resonance there is no energy exchange between the source and the LC pair. The capacitance and inductance simply exchange their energies internally. The only energy that the source needs to supply is that dissipated by the resistance.

We now wish to find an energetic interpretation for the quality factor \( Q \). To this end, let us consider the energy dissipated per cycle at resonance, which we denote \( W_{R\text{(cycle)}} \) and find by dividing the energy dissipated in one second by the number of cycles contained in one second. The former is the maximum average power \( P_{\text{max}} \), and the latter is the resonance cyclical frequency \( f_0 = \frac{\omega_0}{2\pi} \). At resonance we thus have \( W_{R\text{(cycle)}} = P_{\text{max}} f_0 \), or

\[
W_{R\text{(cycle)}} = \frac{2\pi}{\omega_0} \frac{V_{rms}^2}{R} \tag{13.17}
\]

Dividing Equation (13.16) by Equation (13.17) pairwise we obtain \( (W_L + W_C)/W_{R\text{(cycle)}} = Q/2\pi \), which allows us to write

\[
Q = 2\pi \frac{\text{maximum stored energy}}{\text{energy dissipated per cycle}} \tag{13.18}
\]

We thus have yet another interpretation for \( Q \). Since it is based on energy considerations alone, this relation is applicable not only to electrical circuits, but also to nonelectrical systems such as acoustical and mechanical systems. It is thus generally regarded as the basic definition of the quality factor.

In summary, we have three different ways of defining \( Q \):
The definition of Equation (13.18), based on energy considerations.

(2) The definition of Equation (13.10), $Q = \omega_0 / BW$, based on frequency response considerations.

(3) The definition of Equation (13.4), based on circuit elements and topology. As we shall see in the next section, changing the topology from series to parallel changes the expression for $Q$ in terms of the circuit components. However, the meanings of $Q$ in terms of energy or frequency response remain the same.

**Exercise 13.4** Find the maximum stored energy and the energy dissipated per cycle in the circuit of Exercise 13.3.

**ANSWER** 1.267 µJ, 79.58 nJ.

### 13.2 PARALLEL RESONANCE

We now turn our attention to the case of a capacitance and inductance connected in parallel. Once the remainder of the circuit is reduced to its Norton equivalent, we end up with the parallel RLC configuration of Figure 13.4(a). The admittance seen by the source is the sum of the three admittances, $Y = Y_R + Y_C + Y_L$, or

$$Y = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

(13.19)

where $G = 1/R$. This expression is similar to that of Equation (13.1), provided $Z$ and $R$ are replaced with their reciprocals $Y$ and $G$, and $L$ and $C$ are interchanged with each other. Of course, this could have been anticipated using duality considerations. As in the series RLC case, we expect a peculiar behavior when the susceptances $\omega C$ and $-1/\omega L$ cancel each other out.

Retracing similar steps to the series RLC case, we can readily express the admittance of the parallel RLC circuit as

$$Y = G\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

(13.20)

where

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

(13.21)

is the resonance frequency, and

$$Q = R\sqrt{C/L}$$

(13.22)
is the quality factor of the parallel RLC circuit. Note that the expression for $\omega_0$ is the same as for the series RLC circuit, but the expressions for $Q$ are reciprocal of each other, another reflection of the duality principle.

We now wish to find the voltage response $V = V_m/\Theta_v$ to the applied current $I = I_m/\Theta_0$. By Ohm’s Law, $V = ZI = (1/Y)I$. Using Equation (13.20) with $G = 1/R$, and applying Phasor Rule 6 of Section 11.1, we get

$$V_m(\omega) = RI_m \frac{1}{1 + Q^2 \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2} \quad (13.23a)$$

$$\Theta_v(\omega) = -\tan^{-1} Q \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \quad (13.23b)$$

The plot of $V_m$ versus $\omega$ is shown in Figure 13.4(b), where we see that the peak value is $V_{m(max)} = RI_m$. Physically, voltage approaches zero at both ends of the frequency range because of the short-circuit action by $L$ at low frequencies and by $C$ at high frequencies. In fact, $L$ dominates for $\omega < \omega_0$, making the parallel RLC circuit inductive, and $C$ dominates for $\omega > \omega_0$, making it capacitive.

At the borderline frequency $\omega = \omega_0$, the susceptances $\omega C$ and $-1/\omega L$ cancel each other out in Equation (13.19), making $Y$ purely conductive, $Y = G + j0$. Even though the individual admittances $Y_L$ and $Y_C$ are nonzero, their sum vanishes at resonance, making the parallel combination of $L$ and $C$ effectively behave as an open circuit.

It is readily seen that since $V_m$ is maximized at resonance, so is the average power dissipated in the resistance,
The frequencies at which $P$ is down to half its maximum value are the half power frequencies $\omega_L$ and $\omega_H$, and the half-power bandwidth, $BW = \omega_H - \omega_L$, is such that

\[ Q = \frac{\omega}{BW} \]  

(13.25)

Combining Equations (13.21) and (13.22), we readily find that

\[ BW = \frac{1}{RC} \]  

(13.26)

Equations (13.21) and (13.26) indicate that in a parallel RLC circuit $\omega_o$ is set by $L$ and $C$, and $BW$ is set by $R$ and $C$.

**Exercise 13.5** A parallel RLC circuit has $R = 20 \, k\Omega$, $BW = 10^{-4} \, \text{rad/s}$, and $Q = 100$. Find $L$, $C$, and $\omega_o$.

**ANSWER** 200 $\mu$H, 5 nF, $10^6 \, \text{rad/s}$.

Following the line of reasoning of the previous section, it is straightforward to show that in a parallel RLC circuit we have

\[ I_R = \frac{V_I}{\mathbb{0} / R} \]  

(13.27a)

\[ I_C = \frac{\omega C V_I}{\mathbb{0} + 90^\circ} \]  

(13.27b)

\[ I_L = \frac{(1/\omega L)V_I}{\mathbb{0} - 90^\circ} \]  

(13.27c)

Moreover, at resonance, the following identities hold:

\[ V = RI_o / 0^\circ \]  

(13.28)

\[ I_R = I_o / 0^\circ \]  

(13.29a)

\[ I_C = QI_o / 90^\circ \]  

(13.29b)

\[ I_L = QI_o / -90^\circ \]  

(13.29c)

**Exercise 13.6** Derive Equations (13.27) through (13.29).

**Exercise 13.7** In the circuit of Figure 13.4(a) let $I_m = 1 \, \text{mA}$, $L = 10 \, \text{mH}$, and $C = 10 \, \text{nF}$. Find the value of $R$ that yields $Q = 1$. Hence, sketch and label the phasor diagrams for the cases $\omega = \omega_o / 2$, $\omega = \omega_o$, and $\omega = 2\omega_o$. Be neat and precise.

**ANSWER** 1 k$\Omega$. 

\[ P_{\text{max}} = \frac{1}{2} R I_m^2 = R I_{\text{rms}}^2 \]  

(13.24)
We again note that if $Q > 1$, the peak amplitudes of the currents through the reactive elements exceed the peak amplitude of the applied source, a phenomenon known as **resonant current rise**. The amount of magnification is $Q$. Comparing the expressions for the quality factors, we note that in a *series RLC* circuit the smaller the resistance the greater the resonance voltage rise; in a *parallel RLC* circuit, the larger the resistance, the greater the resonance current rise. Can you justify this physically?

The maximum stored energy at resonance, $W_L(t) + W_C(t)$, and the power dissipated per cycle at resonance, $W_{R(cycle)}$, can be obtained from the series RLC expressions of Equations (13.16) and (13.17) using the duality principle. Thus, replacing $V_{rms}$ with $I_{rms}$ and $1/R$ with $R$ in these equations yields

\[
W_L(t) + W_C(t) = \frac{Q}{\omega_p} R I_{rms}^2 \tag{13.30}
\]

\[
W_{R(cycle)} = \frac{2\pi}{\omega_p} R I_{rms}^2 \tag{13.31}
\]

As in the series case, the maximum stored energy is *time invariant*, indicating that at resonance there is no energy exchange between the source and the LC pair; the capacitance and inductance simply exchange their energies internally. The only energy that the source needs to supply is that dissipated by the resistance. As in the *series RLC* case, we have

\[
Q = \frac{2\pi}{\omega_p} \frac{\text{maximum stored energy}}{\text{energy dissipated per cycle}} \tag{13.32}
\]

As mentioned in the previous section, the expressions for $Q$ based on *energy or frequency response* considerations hold regardless of the particular circuit or system. However, the expressions for $Q$ in terms of the *circuit elements* depend on topology, as reflected by the fact that the expressions of Equations (12.4) and (11.22) are *reciprocals* of each other.

**Exercise 13.8** Referring to the *RLC* circuit of Exercise 13.7, find the maximum stored energy and the energy dissipated per cycle, both in J.

**ANSWER** 5 nJ, 31.42 nJ.