Solution

1a) At P distance \( r \) from the line

Gauss' law: \( \oint \vec{D} \cdot d\vec{A} = \Phi_{\text{enclosed}} \)

\[ \varepsilon_0 E \cdot 2\pi rl = \rho l \]

\[ \vec{E} = \frac{\rho}{2\pi\varepsilon_0 r} \]

\( \vec{r} \) is the unit vector on \( \overline{OP} \)

At \( P(0,y,0) \):

\[ \vec{E} = \vec{E}_1 + \vec{E}_2 \]

\[ \vec{E}_1 = \frac{\rho}{2\pi\varepsilon_0 r_1} \vec{\hat{r}}_1 \]

\[ \vec{E}_2 = \frac{\rho}{2\pi\varepsilon_0 r_2} \vec{\hat{r}}_2 \]

\( \vec{r}_1, \vec{r}_2 \) are unit vectors on \( \overline{C_1P} \) and \( \overline{C_2P} \)

\[ \vec{n} = \vec{r}_2 - \sqrt{y^2 + d^2} \]

\[ E = 2E_1 \cos \theta = 2E_1 \cdot \frac{y}{\sqrt{y^2 + d^2}} = \frac{\rho l}{2\pi\varepsilon_0 \sqrt{y^2 + d^2}} \cdot \frac{y}{\sqrt{y^2 + d^2}} \]

\[ \vec{E}_2 = \frac{\rho l y}{\pi \varepsilon_0 (y^2 + d^2)} \vec{\hat{j}} \]

\( \vec{\hat{j}} \) is the unit vector on \( \overline{Oy} \)

(10 marks)

2) a) Choose the Gauss surface which is a cylinder of radius \( r \) (\( a < r < b \))

\[ C = \frac{Q}{V} = \frac{\int_{S} \varepsilon \vec{E} \cdot d\vec{A}}{V} - \int_{V} \varepsilon \vec{E} \cdot d\vec{A} \]

Where \( +Q \) and \( -Q \) are the charges accumulated on the inner and outer cylinders for a given voltage \( V \).
Gauss' law: \[ \varepsilon E = \frac{Q}{2\pi rl} \]

in the insulator

The electric field strength at a point distant \( r \) from the axis: \( E = \frac{Q}{2\pi r l} \). \( \hat{r} \) is the unit vector normal to the cylindrical surface \( 2\pi r \) and directed from the inner to outer cylinder.

\[ V = -\int_{l}^{a} E \, dl = -\int_{l}^{a} \frac{Q}{2\pi l} \, dr = -\frac{Q}{2\pi l} \ln \frac{b}{a} \]

\[ V = \frac{Q}{2\pi l} \ln \frac{b}{a} \]

Thus, \[ C = \frac{Q}{V} = \frac{2\pi \varepsilon l}{\ln \frac{b}{a}} \] is the capacitance of the cable (5 marks).

The leakage current flowing through the insulator between two cylinders is \[ I = \int_{S} \sigma E \, da \]. The insulation resistance is

\[ R = \frac{V}{I} = \frac{\int_{S} E \, da}{\int_{S} \sigma E \, da} \]

As \( \int_{S} \sigma E \, da = \sigma \frac{Q}{\varepsilon} \), \( 2\pi rl = \varepsilon \frac{Q}{\varepsilon} = I \)

\[ R = \frac{V}{I} = \frac{V}{\frac{Q}{2\pi l}} \ln \frac{b}{a} \]

(5 marks)

Alternatively:

\[ dR = \frac{1}{\sigma} \frac{dr}{2\pi rl} \]

\[ R = \int_{a}^{b} dR = \int_{a}^{b} \frac{V}{2\pi l} \frac{dr}{\ln \frac{b}{a}} \]

\[ = \frac{V}{2\pi l} \ln \frac{b}{a} \]
\[ T_1 = RC = \frac{Vt}{2\pi} \ln \frac{b}{a} - \frac{2\pi l}{\ln \frac{b}{a}} = \frac{E}{a} \]

This can be obtained directly from (x) and (**).

\[ T_1 = \frac{-\int E \cdot d\alpha}{\int_0^{\theta_0} E \cdot d\alpha} = \frac{E}{a} \quad \text{independent of } a, b, \text{ and } l. \]

\[ [T_1] = [RJ], [C] = \frac{[V]}{[I]} \quad \frac{[Q]}{[Q]} = \frac{[Q]}{[Q]} = \text{sec.} \quad (5 \text{ marks}) \]

3. a) Choose a circle going through P with centre O located on the axis. Ampère's law gives:

\[ \oint H \cdot dl = I \]

As \( TB = \mu H \), \( \frac{B}{\mu} \cdot 2\pi r = I \)

Magnetic flux density at point P distant \( r \) from the axis: \( a < r < b \)

\[ B' = \frac{\mu I}{2\pi r}, \quad \hat{m} \text{ is the unit vector normal to } CP \text{ and directed into the paper's plane.} \quad (5 \text{ marks}) \]

b) Consider a cylinder of radius \( r \) and thickness \( dr \) of the insulating material.

\[ d\phi = B' \cdot d\hat{m} = \frac{\mu I}{2\pi r} r 

\]
The flux \( \Phi \) through the insulator of the cable is
\[
\Phi = \int_S \mathbf{B} \cdot d\mathbf{a} = \int_a^b \frac{\mu_0 I l}{2\pi r} \, dr = \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a}
\]
The inductance of the cable is:
\[
L = \frac{\Phi}{I} = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}
\]
(5 marks)

\[
T_2 = \frac{L}{R} = \frac{\mu_0 I}{2\pi} \frac{\ln b}{a} \cdot \frac{2\pi h l}{\ln b} = \mu_0 h l^2
\]
(5 marks)

\[
\begin{bmatrix} T_2 \\ R \end{bmatrix} = \begin{bmatrix} L \\ R \end{bmatrix} = \begin{bmatrix} \Phi \\ I \end{bmatrix}, \quad \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} \Phi \\ V \end{bmatrix} = \sec
\]

4. a) Consider a strip of thickness \( dy \) located at \( y \), and choose a circle going through the strip with centre located on the \( z \)-axis.

Ampère's law gives:
\[
\oint \mathbf{B} \cdot d\mathbf{s} = i(t)
\]

\[
A e^{-\frac{B^2}{\mu_0}} \rightarrow B = \frac{\mu_0}{2\pi y} i(t) = \frac{\mu_0}{2\pi y} I_m \cos(\omega t)
\]

The magnetic flux density at a point of the strip:
\[
\mathbf{B} = -i \left[ B_m \cos(\omega t) \right], \quad B_m = \frac{\mu_0 I_m}{2\pi y}
\]

\[
\mathbf{d\Phi} = \mathbf{B} \cdot d\mathbf{a} = \left[ -i B_m \cos(\omega t) \right] \left[ i \, dy \right] = \left[ i B_m \cos(\omega t) \right] dy
\]

It is the unit vector on \( X \)-axis.

Magnetic flux through one turn of the coil.
\[ \phi = \int \mathbf{B} \cdot d\mathbf{A} = \int \mathbf{a} \cdot \mathbf{B}_m (\cos \omega t) \, dy \]
\[ = a \cos \omega t \int_a^{2a} \frac{\mu_0 I_m}{2\pi y} \, dy \]
\[ = \left( \frac{\mu_0 I_m a \ln 2}{2\pi} \right) \cos \omega t = \phi_m \cos \omega t \]

The flux linkage through the coil due to \( i(t) \) is
\[ \lambda = N\phi = N\phi_m \cos \omega t \quad \phi_m = \frac{\mu_0 a \ln 2}{2\pi} I_m \]

The induced emf:
\[ e = -\frac{d\lambda}{dt} = -N\phi_m \frac{d\cos \omega t}{dt} = N\phi_m \omega \sin \omega t \]
\[ e = E_m \sin \omega t \quad (V) \quad E_m = \frac{N\omega \mu_0 a \ln 2}{2\pi} I_m \quad (V) \]

(10 marks)

b) \[ i_R(t) = \mathcal{E}(t) \]
\[ = \frac{E_m}{r+R} \]
\[ = \frac{Em}{r+R} \sin \omega t \quad (A) \]

The direction of \( i_R(t) \) is such that it opposes the change of the flux linkage, i.e., it reduces the total current in the Ampere law circle: \( i_R \) direction is given in the Figure.

Its RMS value \[ I_R = \frac{E_m/\sqrt{2}}{r+R} \]

(10 marks)
5) a) \[ V = \frac{V}{Z_L} \]
\[ I_L = \frac{\hat{V}}{Z_L} = \frac{240 V}{8 + j 6} = \frac{240 Lc}{10 L36.9^\circ} = 24 L36.9^\circ \text{ A} \]
\[ I = I_L = 24 \text{ A} \]
\[ P = R_L I^2 = (8 L2L) I^2 = 8 \cdot 24^2 = 4608 \text{ W} \]
\[ \text{PF} = \cos \phi = 0.8 \] (10 marks)

b) To correct the PF to unity, the capacitor current \( \hat{I}_C \) should lead the voltage \( \hat{V} \) by 90\(^\circ\) and have a magnitude \( \hat{I}_C = \frac{V}{\omega C} \) such that the total current \( \hat{I} = \hat{I}_L + \hat{I}_C \) is in phase with the voltage \( \hat{V} \).

Or: \[ I = I_L \cos \phi = 24 \cdot 0.8 = 19.2 \text{ A} \]
\[ I_C = I_L \sin \phi = 24 \cdot 0.6 = 14.4 \text{ A} \]
\[ \Rightarrow C = \frac{V}{2\pi f I_C} = \frac{240}{2 \cdot 3.14 \cdot 50 \times 14.4} = 0.0531 \text{ F} = 53.1 \text{ mF} \]

The phasor diagram is shown below.

(10 marks)

6) The magnetic circuit is shown below, where
\[ Fm = NI = 350 \times 1.2 = 420 \text{ A-turns} \]
Magnetic reluctance of the core
\[ R_c = \frac{L_c - L_g}{\mu_c A_c} = \frac{L_c}{\mu_c A_c} = \frac{40 \times 10^{-2}}{5.1 \times 4\pi \times 10^{-7} \times 16 \times 10^{-4}} = 3979 \, \text{A/Wb} \]

of the air gap
\[ R_g = \frac{L_g}{\mu_0 A_g} = \frac{0.5 \times 10^{-3}}{4\pi \times 10^{-7} \times 16 \times 10^{-4}} = 248,680 \, \text{A/Wb} \]

Magnetic flux
\[ \Phi = \frac{F_m}{R_c + R_g} = \frac{420}{3979 + 248,680} = 1.66 \times 10^{-3} \, \text{Wb} \]

Flux linkage
\[ \lambda = N \Phi = 350 \times 1.66 \times 10^{-3} = 0.58 \, \text{Wb} \]

\[ \text{Magnetic flux density} \]
\[ B = \frac{\Phi}{A_c} = \frac{1.66 \times 10^{-3}}{16 \times 10^{-4}} = 1.04 \, \text{T} \]

7a) The equivalent circuit of a single-phase transformer

\[ V_2 = n \, V_1 \]
\[ I_2 = I_1 \]
\[ Z_L' = n^2 \, Z_L \]
\[ (R_2 + jX_{L2}) = n^2 (R_2 + jX_{L2}) \]

\[ n = \frac{N_1}{N_2} : \text{transformer ratio} \]

b) \[ Z_L = R_L + jX_{L1} \] is neglected : \[ V_m = V_1 = 120 \, \text{V} \]
\[ I_e = 5 \, \text{A} \implies S_e = V_m I_e = 120 \times 5 = 600 \, \text{VA} \]
\[ Q_e = \sqrt{S_e^2 - P_e^2} = \sqrt{600^2 - 180^2} = 572 \, \text{VA} \]
\[ P_c = R_c I_c^2 = R_c \left( \frac{V_m}{R_c} \right)^2 = \frac{V_m^2}{R_c} \Rightarrow \]

\[ R_c = \frac{V_m^2}{P_c} = \frac{120^2}{180} = 80 \, \Omega \]  

Similarly

\[ X_m = \frac{V_m^2}{\Delta_c} = \frac{120^2}{572} = 25.2 \, \Omega \]

(10 marks)

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